A Common Sense Introduction to Logic

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Contents

Preface

This text was written for an introductory logic course at Baruch College, City University of New York. One goal of this text is to explain the core concepts behind propositional symbolic logic in terms that are as informal and conversational as possible. Given that logic is a predominantly formal field of inquiry, this might seem like a strange approach. However, formalism can be daunting to many students. Formal methods are meant to impart clarity and precision into a given discourse - but for many students, if they are presented with formal explanations right off the bat, it can often have the opposite effect. This is rather an abuse of formalism, and an obstacle to clarity and understanding.

We do not believe that it is necessary to sacrifice rigor or accuracy in order to present this material in accessible terms. However, unlike many texts in symbolic logic, formal definitions and theorems are not the primary focus of this text. Of course, given that this this text covers symbolic logic, we cannot do away with formalism. But it's crucial that the formalism be genuinely understood, and not merely memorized by rote.

Hence, we believe that an intuitive understanding of logic and logical concepts is as important for students as, e.g., their ability to memorize natural deduction inference rules.

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Part I

Informal Logic & Logical Concepts

Chapter 1

Introduction to Logic

1.1 What is logic?

There are many ways to define logic. Some say that logic is “the science of truth,” while others say that it is about understanding how one ought to think. These are very lofty definitions, and they may make it hard for students to understand what it really means to “do logic” in an introductory classroom setting. There is no single correct way to define logic, but we will try to present logic in more concrete terms.

For the purposes of this class, we will understand **logic** to be **the study (analysis, evaluation, and criticism) of *arguments***.

1.2 What is an argument?

In everyday life, we think of an argument as the kind of thing that usually happens when you accidentally hit a car’s bumper, or step on someone’s foot – two people shouting at each other and being angry until one of them gets their way. But in logic, we think of arguments in a different sense.

An argument in logic is more like an argument in the legal sense, where, for instance, a defense attorney will present an *argument* to the jury to try to *convince* them by way of *reasons* that their client is innocent. So, an argument is a series of sentences that are intended to provide reasons to support a conclusion.

That sounds very general, and indeed it is, but note that not all bits of language make up an argument. For instance, if someone is telling you about everything that happened to them on their way home from work that day, that probably isn’t an argument in the sense that we’re talking about. It’s more of a *report*. Stating some facts or describing a series of events is not an argument if the person isn’t trying to *convince* you of something, or doesn’t have a “point to make.”

1.3 What is an entailment?

Suppose you wanted to convince someone of something that they don't already believe. How could you do it?

There are many ways to get someone to agree with you that are not relevant to logic. For instance, you could entice them with money (bribery); threaten them with a baseball bat (coercion); or, feed them a mind-altering drug that would make them accept your belief (brainwashing). These are not really "fair" ways to win an argument, though.

In making a logical argument, the strategy is rather different. Suppose you want to convince someone to accept some statement – let's call the statement Z (as you will soon see, we often use letters as placeholders for sentences in logic). They don't believe Z, so what can you do? Well, the strategy that is employed in logic is to start with some statements that the person *does* believe – say, A, B, and C – and then show them that *if* they believe A, B, and C, then they *have to* accept Z!

In what sense do they "have to" accept Z? The technical notion that we will explore is that A, B, and C together **imply** or **entail** Z. But it's best to explain by way of example.

Consider the following sentences:

Example (1)
(A) Either Alice or Bob must chair the meeting.
(B) Alice is sick, so she can't chair the meeting.
(Z) Therefore, Bob will chair the meeting.

Now suppose someone accepts both (A) and (B) – they know that only Alice and Bob can chair meetings, and they know that Alice can't chair this meeting because she is sick – but they *don't* accept (Z); that is, they refuse to accept that Bob will chair the meeting. (Let's assume, also, that the meeting is definitely going to happen.)

The reaction you might have to such a person is – what the heck is wrong with them?! How could someone believe (A) and (B) and not believe (Z)? They must be crazy or irrational! Certainly, there appears to be something wrong in the way they are thinking. Thus, it should be clear that there is an intuitive sense in which believing (A) and (B) *forces* you to accept (Z). Failure to do so would constitute a lack of *rationality*, the basic rules that guide human thought.

Here's another way to put the point: If we assume that (A) and (B) are true, could (Z) *possibly, in any way, fail to be true*? The answer is *no*! If (A) and (B) are true, (Z) *must be the case.*

So, there is a very interesting relationship between the sentences (A), (B), and (Z). First, if (A) and (B) are true, (Z) *must necessarily* be true. Second, if anyone *believes* (A) and (B) to be true, then they *must*, or *ought to*, believe that (Z) is true as well, or else we know there is something seriously wrong with their thinking. This relationship – between (A) and (B), and (Z) – is known as **entailment** (or, also, **implication** or **consequence**), and it is one of the central concepts in logic. Because (A) and (B) being true means that (Z) *must* be true, we say that (A) and (B) together **entail** or **imply** (Z) – or, that (Z) is an **entailment/implication/consequence** of (A) and (B).

Here is another example to illustrate the basic point:

Example (2)
(A) Alice is older than Bob.
(B) Bob is 35 years old.
(Z) So, Alice is at least 35 years old.

Suppose that someone believed (A) and (B), but failed to believe (Z). That is, they believed all of the following:

Example (2*)
(A) Alice is older than Bob.
(B) Bob is 35 years old.
(Z′) Alice is less than 35 years old.

It would be hard to reason with such a person, to say the least. For they seem to believe that Alice is older than Bob, but also that Alice's age is less than Bob's age. What is going on?? This shows once again that because (A) and (B) together imply or entail (Z), it would be irrational to believe (A) and (B), but not (Z).

Logic is very interested in understanding this special relationship between sentences that holds when one set of sentences (like (A) and (B)) entails another (like (Z)).

1.4 The Structure of a Logical Argument

We can now understand the basic structure of a logical argument, and the basic strategy used in making rational arguments.

We have seen that when one set of sentences entails another sentence, it would be irrational to accept the first set of sentences and not accept the other one. So, suppose you wanted to convince someone of some proposition (e.g., *Alice is at least 35 years old*) that they didn't believe. In logical argumentation, you start with some statements that they *will* accept (e.g., maybe they already believe that Alice is older than Bob, and that Bob is 35 years old, but they just haven't "put everything together"), and then show that those statements *entail* the proposition that you want them to accept. So, if your reasoning is good, your opponent will be *forced* to accept your conclusion if they are a reasonable person. (Unfortunately, people are not always reasonable!)

Thus, the basic structure of a logical argument is as follows:

- The point of an argument is to prove some statement or claim. This claim (or, perhaps, set of claims, but usually just one) is known as the conclusion. The conclusion is the main goal, target, or purpose of the argument. In the above example (2), the **conclusion** is that Alice is at least 35 years old.

- According to the strategy explained above, in order to prove the conclusion, you have to start with some statements that are intended to be accepted by both parties. These are known as the **premises** of the argument. They are the "raw material" that is used to build up the argument. Since you can't provide a reason for *every single claim* that you are going to make in your argument (think of the endless children's game of asking *Why?*), there have to be some claims that are just assumed, or taken for granted (at least for the purposes of the argument). So, every argument has **premises** or **assumptions** that are not themselves argued for. However, if your opponent disagrees with one of your premises, then they will challenge your argument, and you may have to provide reasons for the challenged premise.

- Once you have your premises, then you use logical principles to combine the premises in certain ways and make inferences that will eventually lead to your conclusion. If you make a sophisticated inference to a claim that was not obvious from the premises, you might describe that claim as an **intermediate conclusion** – it's something that you have argued for (so, in that sense, it's not a premise), but it's not the *main* conclusion that the argument is working towards. Not every argument has an intermediate conclusion.

Here is another example argument illustrating these concepts:

Example (3)

(A)	Carol is 5'5" tall.	(PREMISE)
(B)	Alice is taller than Bob.	(PREMISE)
(C)	Bob is taller than Carol.	(PREMISE)
(D)	So, Alice is taller than Carol.	(INFERENCE / INTERMEDIATE CONCLUSION)
(Z)	Therefore, Alice is over 5'5" tall.	(CONCLUSION)

Notice that we *infer* that (D) *Alice is taller than Carol* on the basis of the premises that Alice is taller than Bob and Bob is taller than Carol. (For it's universally true that if person A is taller than person B, and person B is taller than person C, then person A is taller than person C – i.e., *being taller* than is a *transitive property*.) So, since we aren't immediately given the fact that Alice is taller than Carol, we must make an inference to that claim. Then we can infer that Alice is over 5'5" tall, because we have shown that she is taller than Carol, who is 5'5" tall. Because (D) is the result of inference or deduction, it qualifies as an intermediate conclusion. It is not a premise because premises are supposed to offer new, independent pieces of information, whereas (D) is simply connecting the information that is contained in (B) and (C).

Pay attention to the fact that even though it's kind of *obvious*, we still have to state explicitly that (D) *Alice is taller than Carol*, in order for the argument to be complete and logically acceptable. For example, the following argument does not work:

Example (4)

(A)	Carol is 5'5" tall.	(PREMISE)
(B)	Alice is taller than Bob.	(PREMISE)
(C)	Bob is taller than Carol.	(PREMISE)
(Z)	Alice is over 5'5" tall.	(CONCLUSION)

Strictly speaking, this argument is incomplete because all we are explicitly told is that Alice is taller than Bob, and that Carol is 5'5" tall - we need to explicitly connect the three pieces of information contained in the premises in order to reach the conclusion. You might think, *We can just infer directly that Alice is over 5'5" tall because it's obvious that she's taller than Carol, who is 5'5" tall.* But the fact that Alice is taller than Carol is not explicitly one of the premises. It has to be inferred from (B) and (C). So, in constructing a logical argument, it is essential to make *every step of reasoning* explicit. Sophisticated logical arguments are built up by making small steps that gradually lead towards the conclusion. Every step must be explicit so that it is easy to see if the argument employs good reasoning or not.

1.5 Key Words

Consider the following sentences:

(5) It's going to rain.

(6) Therefore, it's going to rain.

Notice the effect that the word 'therefore' has on sentence (6). It doesn't really change the information that is conveyed by (5) - rather, it suggests that we are dealing with the conclusion of an argument. So the function of 'therefore' is to signal to the reader that the sentence is supposed to be a conclusion. Thus, it provides information about the logical structure of the text.

There are many words in English that signal logical structure. Being aware of these words makes it easier to distinguish premises and conclusions when analyzing an argument. Here we will review some of these logical key words:

CONCLUSION:

- So
- Hence
- Therefore
- Thus
- Consequently
- As a result
- It follows that

PREMISE:

- Because
- Since
- Due to the fact that
- As
- Given that
- Furthermore
- In addition
- Besides

(Note that there is no reliable indicator that differentiates the main conclusion from intermediate conclusions. Instead, you should rely on your intuitive grasp of the purpose of the text.)

Some of the premise key words can be slightly confusing at first, because they only work in complex sentences:

Example (7)
(A) Pluto is a dog.
(B) Since all dogs are fluffy, (C) Pluto is fluffy.

In this example, (A) and (B) are premises, and (C) is the conclusion. The word 'since' indicates that the fact that all dogs are fluffy is supposed to provide a reason for believing that Pluto is fluffy. Since reasons/justifications are typically *premises*, we can infer that 'since' is a premise indicator. However, due to the grammar of 'since,' it can only appear as part of a complex sentence, where the first part provides the premise, and the second part offers the conclusion. That is why we have split the second sentence into two parts - each part has a different logical purpose in the argument. This kind of pattern occurs with respect to 'Because,' 'as,' 'due to the fact that,' and 'given that,' as well:

Example (8)
(A) If it rained, the grass should be wet.
(B) Because the grass isn't wet, (C) it must not have rained.

Example (9)
(A) I'll either go to the movies or watch TV.
(B) Due to the fact that my TV is broken, (C) I'll go to the movies.

Although one can usually recognize the structure of a logical argument without relying on such indicators, being aware of these keywords may simplify the task of argument analysis.

1.6 Factual vs. Practical Arguments

The study of logic is concerned with the *evaluation* of rational arguments. But when you evaluate something, you have to know what *type* of thing it is. For instance, an ugly statuette might be a bad *piece of art* but a good *paperweight*. However, if it was intended to be a piece of art, then we ought to evaluate it as art, and not as a paperweight. Similarly, there are different **types of arguments**, to which different concepts and criteria apply. Thus, it is important to be able to identify argument types.

Recall that an argument is a set of sentences intended to provide reasons for accepting a conclusion. Sometimes, the conclusion is that *something is true*, or *is the case*, or is a fact. The following argument, as well as the examples discussed in previous sections, is of this type:

Example (10)
(A) If that's Alice's coat, then she must be tall.
(B) That is Alice's coat.
(Z) Therefore, Alice is tall.

The claim, *Alice is tall*, is a *factual* claim – it's either true or false, it can be decided by measuring Alice's height with a tape measure, etc. It's also not a mere matter of subjective opinion – there really is a fact of the matter as to whether Alice is tall.

(You might be wondering: what if Alice is 5'9" tall? Is the statement 'Alice is tall' true or false in that case? It may be hard to decide. This reveals a very messy problem in logic – vagueness*. It may actually be impossible to know exactly what conditions need to be satisfied for Alice to count as being tall, so the statement that 'Alice is tall' seems like it might be* neither true nor false *if Alice is on the borderline. This is a Seriously Big Problem, but we will ignore it for simplicity. For the purposes of classical logic, we simply assume that every statement is* either *true or false. So, we will pretend that vagueness does not exist.)*

So, if the conclusion of an argument is a statement of fact, then we will say that it is a **factual argument**. However, some arguments are not intended to provide reasons for something being *true*, but rather are intended to provide reasons for *what someone should do.*

Example (11)
(A) We ordered delivery for dinner last night.
(B) Ordering delivery is more expensive than cooking.
(Z) Therefore, we should cook dinner tonight.

Consider the conclusion of this argument: *we should cook dinner tonight.* Is there really a fact of the matter about whether or not they should cook dinner? If they're trying to save money, then maybe that would be the best thing to do. But if they don't have enough time to prepare food, then maybe they should order in.... There are pros and cons to either decision. But the point is that this argument is intended to provide reasons for *making a decision*, not reasons that a certain statement or claim is true.

If we have an argument whose conclusion is a *decision*, or that some action *should* be performed, then we will call it a **practical argument**. Here is another example:

Example (12)
(A) Paris is nice at this time of year.
(B) French food is delicious.
(C) There are cheap flights to Paris available.
(Z) So, I should take a vacation to Paris.

Notice that (A), (B), and (C) do provide reasons for the conclusion, (Z). But, notice also that they do not *entail* the conclusion. A person isn't *rationally required* to agree that they should take a vacation to Paris, even if they agree with (A), (B), and (C) - in which case, this is a very different sort of argument than (10).

1.7 Inductive vs. Deductive Arguments

The subject of practical reasoning (reasoning about practical arguments) is of great philosophical interest, but in this course we will be exclusively focused on factual arguments. Furthermore, we will mostly be focused on a specific kind of factual argument – deductive arguments.

Deductive and inductive arguments differ in *the kind of support that the argument is supposed to provide for the conclusion.* In a deductive argument, the premises are meant to provide conclusive, indisputable reasons for accepting the conclusion: if the premises are the true, the conclusion *must* be true, and there's no possible way for it to be false. Recall our previous example:

Example (2)
(A) Alice is older than Bob.
(B) Bob is 35 years old.
(Z) Alice is at least 35 years old.

If (A) and (B) are true, there is simply no possible way for (Z) to be false – it would be a *contradiction* to accept (A) and (B) and deny (Z). Thus, (A) and (B) are meant to provide definitive reasons for accepting (Z). This is a sign of a **deductive argument**.

An **inductive argument**, on the other hand, is one in which the premises are intended to provide *evidential support* for the conclusion, but not definitive proof. Consider:

Example (13)
(A) Most basketball players are very tall.
(B) Carol is a basketball player.
(Z) Therefore, Carol is very tall.

In some sense, this is a good argument. (A) and (B) do provide a reason to think (Z) is likely, and if (A) and (B) are true, then (Z) is probably true as well. But notice that, unlike with argument (2), if (A) and (B) are true in (13), it isn't *necessarily* the case that (Z) is true. Some basketball players are not very tall, and maybe Carol is one of those players who happens to be not-very-tall. So, since it's possible that (A) and (B) are true, but (Z) is false, (A) and (B) do not entail (Z), i.e., they do not provide absolute, definitive proof of (Z). However, since they do provide *good evidence* of (Z), Argument B is still a good *inductive* argument.

Thus, an **inductive argument** is one in which the premises are intended to provide *evidence* for the conclusion, but not wholly definitive proof. Consider another example:

Example (14)
(A) Robins are birds that fly.
(B) Sparrows are birds that fly.
(C) Finches are birds that fly.
(Z) Therefore, all birds can fly.

Once again, this is not such a bad inductive argument. However, although the premises are true, the conclusion is false, since *not all* birds fly (e.g., penguins). Thus, Argument C is not very good when considered as a deductive argument. The premises make the conclusion *likely* to be true, but they do not entail the conclusion.

You may notice that inductive arguments often have a similar flavor – X, Y, and Z have some property (flying, being tall, etc.), therefore *other* things that are like X, Y, and Z will *also* have that property. Inductive reason typically involves using information about a specific subset of entities, and then extrapolating or generalizing that information to other similar entities. This is a good way to reason, and it is essential for natural science, but it is very different than deductive reasoning.

Although the study of inductive reasoning has grown by leaps and bounds, it is still less well understood than deductive reasoning. Probabilistic reasoning has seen a boon in recent years. Nevertheless, deductive reasoning is much more well-understood and will be the focus of this course.

KEY CONCEPTS

- Entailment/Implication
- Premise
- Conclusion
- Intermediate Conclusion
- Logical Keywords
- Factual vs. Practical Arguments
- Deductive vs. Inductive Arguments

RECOMMENDED EXERCISES

- Argument Analysis:
  → http://logic.baruchsites.com/exercises/argument-analysis
- Factual vs. Practical Arguments:
  → http://logic.baruchsites.com/exercises/factual-vs-premlctical
- Deductive vs. Inductive Arguments:
  → http://logic.baruchsites.com/exercises/deductive-vs-inductive

Chapter 2

Validity, Soundness, Strength, Cogency

2.1 Validity and Soundness

We will now approach perhaps the most important concept of this unit, and one of the most important notions in logic – **validity**.

We have already begun to circle around this idea in our discussion of entailment. Recall this argument:

Example (2)
(A) Alice is older than Bob.
(B) Bob is 35 years old.
(Z) Alice is at least 35 years old.

We said that in (2), the premises *entail* the conclusion in virtue of the fact that (a) if the premises are true, the conclusion *must necessarily* be true, and (b) if you accept the premises, you are rationally obligated to accept the conclusion.

A deductive argument in which the premises entail the conclusion is known as a **valid argument**. Consider another argument:

Example (15)
(A) Either cows are red, or pigs can fly.
(B) Cows are not red.
(Z) Therefore, pigs can fly.

Let's examine this argument. Obviously, it is not an amazing argument because it has an absurd conclusion – pigs can't fly. But is it all bad? Pretend you don't know anything about farm animals; you don't know what cows look like; and you don't know if pigs are birds or fish or mammals. All you know is what is claimed in (A) and (B). *In that situation*, you would have a good reason to accept (Z). In fact, (A) and (B) provide *conclusive* reasons to accept (Z) – just as with Argument A, it would be irrational to accept (A) and (B) and not accept (Z). Similarly, *if* (A) and (B) *were* true (even though we know (A) is false), then (Z) *would necessarily* be true – there is no way for (A) and (B) to be true, but (Z) false. Thus, we can see that although Argument (15) has a false conclusion, it still has that interesting connection between the premises and the conclusion – namely, the premises *entail* the conclusion, and thus the argument is *valid.*

Here is our definition of validity:

Validity

A deductive argument is valid just in case if the premises are true, then the conclusion must be true (i.e., it is impossible for the premises to be true and the conclusion to be false).

Logic is more interested in *reasoning* itself than whatever the reasoning is about (e.g., farm animals, in this case). Thus, when evaluating a deductive argument, we will mostly be focusing on whether or not it is *valid*, disregarding whether the premises are true or false. Notice that when judging validity, we consider what follows *if* the premises are true. Therefore, when discussing validity, we may simply *assume* the premises are true, and then ask ourselves – does the conclusion *follow*?

Validity is a key concept in logic because logic studies patterns of reasoning, and sometimes you can employ good reasoning even if you start from false assumptions. For instance, imagine that you're told some important information about the weather - say, that there's going to be a thunderstorm arriving tomorrow. You use that information to reason that the plants in the garden will get wet, and so they don't need to be watered today. That is a fairly reasonable inference to make from the claim that there is going to be a thunderstorm tomorrow. But suppose there *isn't* going to be a thunderstorm tomorrow - the information you were told was simply false. In that case, you may have arrived at an inaccurate conclusion. However, that doesn't mean that you weren't employing *good reasoning*, albeit on the basis of *false premises.*

Another way to think about validity is in terms of **truth preservation** – when thinking about validity, we don't care whether the premises are true or false, we just assume that they are true and see if they entail the conclusion. Thus, a valid argument is one in which if the premises are true, we can know definitively that the conclusion is true as well – in other words, a valid argument *preserves truth* – if the premises are all true, then truth will be "transmitted" to the conclusion as well.

In sum, validity is an important property of logical arguments which pertains to the *quality of the reasoning* in the argument, and which *ignores* the question of whether the premises are in fact true (the premises are assumed to be true for the purpose of judging validity).

Although logic is *primarily* interested in validity, in "real life," when we judge an argument, we also care about whether the statements are true or false. This is what the notion of **soundness** pertains to. Once you understand validity, soundness is easy to grasp. A **sound** argument is simply a **valid** argument *that* *has true premises.*

Soundness

A deductive argument is sound just in case it is valid *and* all the premises are true.

Thus, argument (15), although valid, is not sound, because premise (A) is not true. The following type of argument is both sound and valid:

(A) Triangles have three edges.
(B) Rectangles have four edges.
(Z) Therefore, rectangles have more edges than triangles.

Let's make some observations to solidify these concepts:

- **A valid argument may have false premises.** When judging validity, we simply *assume* that the premises are true, even if we know they are false.

- **A valid argument may have a false conclusion.** A valid argument must have good reasoning, but if you start from false premises, then you can end up at a false conclusion, even if your reasoning is valid.

- **A sound argument must have true premises.** Soundness is validity *plus* true premises, so this one is self-evident.

- **A sound argument must have a true conclusion.** This follows from the fact that a sound argument must have true premises, and it must be valid. Since validity preserves truth, a sound argument will also have a true conclusion.

- **An argument may be *invalid* even if every statement in the argument is true.** For example, consider the following argument:

  (A) Grass is green.
  (B) Snow is white.
  (Z) Therefore, the ocean is blue.

  All of the above statements are true, but (A) and (B) do not *entail* (Z) – there is no logical connection between them. Therefore, this is not an example of good (valid) reasoning, even though every statement happens to be true.

2.2 Counter-Examples

Recall that when judging validity, we *assume* that the premises are true, and then we consider whether the conclusion *must necessarily* be true. Thus, a valid argument is one in which it is *impossible* for all of the premises to be true *and* the conclusion false.

This means that there is an intuitive way to think about whether an argument is valid or not. First, clear all factual information from your head (or, more realistically, try to suspend belief in all the facts under discussion); then pretend that the information in the premises is all true; then try to extend that scenario into one in which the conclusion is false. If you are able to imagine some hypothetical situation in which the premises are true, and the conclusion is false, then you have come up with a **counterexample** to the argument, and that counts as proof that the argument is *invalid*.

Let's look at the following argument:

Example (16)
(A) All cows are blue.
(B) Bessie is a cow.
(Z) Therefore, Bessie is blue.

Now, we know this argument is unsound, since it has a false premise: (A). It isn't even clear whether the conclusion has a truth value at all, since Bessie is an imaginary cow. But let's apply the procedure just described: Let's forget what we know about cows, and pretend that all cows are blue, and that there is some cow named Bessie. Is there any conceivable way that Bessie could be some color other than blue? It seems not. Thus, since we are unable to find a counterexample, it is reasonable to conclude that the argument is valid.

Now consider another argument:

Example (17)
(A) All cows are blue.
(B) Bessie is blue.
(Z) Therefore, Bessie is a cow.

Let's try to find a counterexample to this argument. Let's pretend that all cows blue, and that there is something or someone called Bessie, and Bessie is blue. Is it possible to imagine a situation in which Bessie is *not* a cow? Sure, there is! Maybe Bessie is a blue fish. The argument does not state that Bessie is a cow, so he/she/it could just be some other blue thing. Any such scenario would constitute a *counterexample* to this argument, and thus demonstrate that it is *invalid*. This is because we have managed to describe a hypothetical scenario in which all the premises are true and the conclusion is false. Thus, we can see that trying to find a counterexample can be an effective way to judge whether an argument is valid.

However, the counterexample method must be used with caution. If you really find a counterexample, then you can definitely conclude that the argument is invalid. But if you can't think of a counter-example, it's hard to know for sure that the argument is valid – counter-examples sometimes are difficult to come up with, and maybe you just haven't thought of the right scenario yet! So, failure to find a counterexample is usually not conclusive proof that an argument is valid. Sadly, you can often be more confident that you've shown an argument to be invalid, than to have shown one to be valid.

2.3 Strength and Cogency

Validity and soundness are used when evaluating *deductive arguments*. **Strength** and **cogency** are similar terms used to discuss *inductive arguments*. Consider the following inductive argument:

Example (18)
(A) Alice's computer has a virus.
(B) Bob's computer has a virus.
(Z) Therefore, everyone in the office's computer has a virus.

This conclusion seems a little too hasty. You shouldn't infer that everyone in the office has a virus just on the basis of two examples. This would be a case of **weak inductive reasoning** – drawing a conclusion on the basis of weak evidence. Now consider this argument:

Example (19)
(A) Alice's computer has a virus.
(B) Bob's computer has a virus.
(C) Carol's computer has a virus.
(D) Dave's computer has a virus.
(E) Elaine's computer has a virus.
(F) Everyone in the finance division's computer has a virus.
(Z) Therefore, everyone in the office's computer has a virus.

Notice that the more examples you can cite of people in the office who have a computer virus, the more persuasive the argument becomes. It seems like less and less of a coincidence if more and more people in the office are found to have a computer virus. It doesn't *necessarily* mean that the conclusion is true - even if you check the computers of 100 employees, it could be the case that the 101st employee got lucky, and *doesn't* have a computer virus. But it's still an example of **strong inductive reasoning** -– reasoning on the basis of a strong body of evidence. The difference between weak and strong arguments has an obvious connection to the scientific notion of *sample size*, for those who are familiar with that term. An argument that appeals to a larger sample size tends to be stronger.

Recall that in deductive arguments, validity pertains to the quality of the reasoning involved – does the conclusion really follow from the premises, assuming the premises are true? With inductive arguments, we also want to be able to evaluate whether the premises provide good support for the conclusion, independently of whether the premises are in fact true or false. Thus, we use the term strong as the inductive analog of valid:

Strength

An inductive argument is strong just in case the premises provide good evidence for the conclusion (i.e., the premises make the conclusion highly probable).

Just as with validity, when judging strength, we ignore whether the premises are in reality true or false, and we just *assume* they are true and see whether they provide good *support* for the conclusion. Thus, (19) is a fairly strong inductive argument, since it appeals to a decent sample size, but we cannot say whether the premises are true or false in reality (since those people do not exist in reality).

(Note that strength is a matter of degree. An argument can be stronger than another argument if it supplies stronger evidence to support the conclusion, but there is no absolute standard of what counts as a "strong" vs. a "weak" argument. This contrasts with validity, which is not a matter of degree - an argument is either valid or it is not.)

Continuing the analogy between inductive and deductive arguments, just as a sound deductive argument is one that is *valid* and has *true premises*, a **cogent inductive argument** is one that is *strong* (the premises provide good evidence for the conclusion), *and* the premises are *in fact true*.

Cogency

An inductive argument is cogent just in case it is strong *and* all the premises are true.

Thus, if we continue the example in (19), if it turns out that Carol's computer in fact does *not* have a virus, then the argument will contain a false premise, and hence would not be cogent. However, it would still be strong, nevertheless. Notice that cogency, unlike strength, is *not* a matter of degree, since the premises are definitively either true or false.

Finally let's make some observations that we can compare with our observations about deductive arguments:

- **A strong argument may have false premises.** When judging strength, we simply assume that the premises are true, even if we know they are false.

- **A strong argument may have a false conclusion.** Like deductive arguments, a strong argument might have a false conclusion if it starts from false premises. Unlike a deductive argument, a strong argument can have a false conclusion *even if it starts with true premises* (strong arguments only make the conclusion *probable*, not *certain*).

- **A cogent argument must have true premises.** Cogency is strength *plus* true premises, so this one is self-evident.

- **A cogent argument does not have to have a true conclusion.** Since inductive reasoning is probabilistic, even strong inductive reasoning can sometimes lead to a false conclusion.

- **An argument may be *weak* even if every statement in the argument is true.** As with deductive reasoning, if an argument does not follow a logical "train of thought," but instead consists of unconnected (but true) statements, then it would not be a strong argument, even if all the statements happened to be truths.

PUTTING IT TOGETHER

The relation between these concepts can be symbolized in a table:

	Premises support the conclusion	**Premises are true and support the conclusion**
Deductive	Valid (the premises, if true, provide definitive proof of the conclusion)	Sound
Inductive	Strong (the premises, if true, provide good evidence for the conclusion)	Cogent

KEY CONCEPTS

- Validity - Truth Preservation
- Soundness
- Strength
- Cogency
- Counterexample

RECOMMENDED EXERCISES

- Validity – Informal Arguments:
  → http://logic.baruchsites.com/exercises/validity-informal

Chapter 3

Logical fallacies

As you recall, logic is interested in the evaluation of arguments. Usually, in order to do this rigorously, we translate natural language sentences into formal notation, and then use formal tools to evaluate the argument. However, this might seem a bit removed from everyday life. In this section, we'll discuss a phenomenon that is all too readily apparent in everyday forms of persuasion – logical fallacies. Fallacies are *bad* patterns of reasoning, and we'll learn to identify various kinds of logical fallacies. By understanding these forms of bad reasoning, and what's wrong with them, it will make it easier to deal with such arguments when you encounter them in the study of logic, or in ordinary discourse.

So, we'll be going through a number of these examples and explaining when they apply. The names of the fallacies do suggest what is wrong with the arguments that they apply to, but you'll still have to do a fair amount of memorization to be able to identify which fallacies apply to which situations.

3.1 Appeal to Authority

First, let's look at an obvious example of a fallacy, just to get started. Imagine a child is trying to prove to his friend that he is the smartest kid in the class. His argument is: "I'm the smartest kid in the class. My dad says so." Is this is a good argument? Unless his dad is Albert Einstein, it's probably not a good argument. Just because your dad says something, doesn't necessarily make it true. So, this is an example of poor reasoning – the premise doesn't really provide good support for the conclusion. Let's look at another common example:

(20) Milk is bad for you. I heard someone say that on the internet.

Again, this is not a very good argument. The person is appealing to some random person that they heard on the internet, and this doesn't provide very strong support for the conclusion. Notice, however, that there is something similar about both of these rather weak arguments. They both involve trying to appeal to some alleged authority as a way of justifying their claims. And, in both cases, the resulting argument is rather weak. So, we will call this an example of a fallacy – specifically, the fallacy known as "appeal to authority." Oftentimes, we will be able to identify these kinds of *patterns* of bad reasoning – reasoning that is repeatedly misapplied in a number of different circumstances. So, as logicians, as critics of arguments, we want to be able to give names to these patterns of bad reasoning.

Appeal to authority is an argument that says that the conclusion must be true because somebody else says that it is true. "Milk is bad for you. I heard someone say that on the internet." This is a fallacy because the mere fact that someone says something on the internet obviously does not make it true.

Now, occasionally, it can be a good thing to appeal to an authority. If the authority is an expert, like a medical doctor, then it might be sensible to listen to their opinion. But when the supposed authority being appealed to is not really knowledgeable about the subject matter, then it becomes a fallacy, which we call appeal to authority, or appeal to ignorant authority.

3.2 Appeal to Force

In Appeal to Force, the arguer attempts to win the argument by attacking or threatening their opponent. For example:

(21) I should be the leader of this group. If anyone disagrees with me, I'll break their legs!

Now, although this might be a pretty good way to win an argument, it obviously is not a *rational argument*. There is no *reasoning* involved as to why the conclusion is true, there's just a threat about what will happen if the arguer doesn't get his way. So, this is clearly bad reasoning from a logical point of view.

3.3 Appeal to Pity

An argument that appeals to pity attempts to convince the opponent by making them feel sympathy or pity for the arguer's position. For instance:

(22) I should be the leader of this group. If I don't get to be the leader, I'll feel so sad and pathetic!

The opponent might be persuaded to agree with the arguer out of sympathy, but that's not really a rational basis for the conclusion. Again, an argument is supposed to provide reasons as to why the conclusion *is* true, not just why someone should go along with the conclusion, e.g., out of pity, or fear. So, in general, trying to win an argument by appealing to the other person's *emotions* is not going to result in a good rational argument (even if it might be effective in practice).

3.4 Subjectivism

A subjectivist argument is one which says that the conclusion is true simply because I think it's true. It's entirely subjective. Whatever I say, must be the case. So, for example:

(23) I should be the leader of this group. I just know I should and I'm always right about these feelings.

This is clearly a fallacy because you can't know something like that to be true just because you feel that it's true. A good argument should provide objective, logical reasons. If the goal of an argument is to persuade some other party who doesn't already accept the conclusion, then the mere fact that the arguer believes the conclusion to be true probably won't sway that person. So, this is another kind of fallacy.

3.5 Straw Man

A straw man is an argument, or a counter-argument, which attempts to attack someone else's views or position, but in doing so distorts and weakens the other person's position, so that instead of responding to the real person, you are responding to a "straw man" – a weak imitation of the real view that you are attacking. So, with the straw man, someone makes an argument with a conclusion like "Global warming exists," for example, Then, a straw man is a counter-argument which attempts to respond to the argument, but actually attacks a distorted or weakened version of the arguer's position. So, in this example, the counter-argument states (correctly), "It doesn't get hotter every single day of the year!" But the person who's defending global warming obviously isn't saying that that's the case – that's not what global warming implies. So the counter-argument is misrepresenting the position that they are attacking. Rather than attacking their real opponent, they are attacking a "straw man" (like, a weak imitation), in order to make their job easier. This is clearly cheating – if you want to have a rational argument with someone, it's important to give a fair and accurate characterization of their side of the argument. Otherwise, you're just not responding to that person's real views at all!

3.6 Slippery Slope

Another really common fallacy that you will see often in everyday life is the slippery slope. A slippery slope argument basically says that if one thing happens, that will send us falling down the slippery slope, and a whole bunch of other things are likely to follow. This is known as a fallacy, because usually the things that are supposed to follow are not actually that likely to happen, even given the initial event. So, for example:

(24) If they let men into the restaurant without a tie on, then pretty soon people will want to eat here in the nude.

The arguer is trying to say that they shouldn't allow men to eat at the restaurant if they're not wearing a tie, because if *that* happens, then the next step is that people will want to eat in the nude. But is that really the case? It sounds like a big leap. So, it's a fallacy because the supposed "slippery slope" doesn't really exist. Slippery slope arguments try to force you to choose between two extremes – it tends to ignore the possibility of a middle ground, since it tries to argue that once you are on that middle ground, you will quickly end up at the other extreme.

This type of reasoning is often applied when considering permissions and bans: "If you permit A, then people will want to do B, C, D, etc." Or: "If they ban A, then pretty soon they'll ban B, C, D, etc." For a concrete example, consider the debate about gun control in the United States. Opponents of gun control argue that if the government enacts a ban on assault rifles, then pretty soon the Second Amendment will be entirely eliminated. Head of the National Rifle Association, Wayne LaPierre, goes so far as to argue that those who support gun control want to "eradicate all individual freedoms."[1] Notice that this argument assumes that moderate gun control is impossible – once you allow *any* limitations on individual freedoms (such as the freedom to buy military-grade weaponry), then *all* individual freedoms will disappear. So, this is a clear example of a slippery slope argument – it assumes that there is no possible middle ground.

Now, in some cases, a slippery slope type argument isn't necessarily a fallacy. For instance:

(25) If Johnny gets the flu, then the rest of the class will get the flu as well.

This is still probably not a great argument, but at least there is some causal reason to believe that the flu could spread from Johnny to the rest of his classmates. So, we tend to call an argument a slippery slope when it really leaps to a conclusion that isn't justified by the initial premises.

3.7 False Alternative (False Dilemma)

When an argument exhibits a false alternative, or false dilemma, the arguer presents their opponent with a choice between two things, and suggests that only one of them is the truth. The problem, however, is that the choices, or alternatives, that are being offered, are not the only possibilities – so it's forcing you to choose between A and B, when there is also the possibility of C, D, E, etc. So, consider:

(26) Either we stop at the next restaurant we see, or I'm going to die of hunger.

The obvious conclusion is that they should stop at the next restaurant, since it would be bad for the person to die of hunger. But this is a false alternative,

[1] https://www.theguardian.com/us-news/2018/feb/22/nra-wayne-lapierre-gun-control-cpac-speech-2018

because it's not the case that the person is going to die of hunger if they don't eat right away. So a false alternative presents a pair of options when there are really other possibilities to consider.

3.8 Ad Hominem

"Ad hominem" means, roughly, "against the person." An ad hominem fallacy is an argument that doesn't try to win by offering good reasons or evidence, but rather directly attacks the personal *character* of the other person. For example:

(27) I didn't cheat on the exam! You're stupid and ugly!

You might notice this kind of fallacy coming up a lot with political advertisements, since these often appeal to some character flaw in their opponent. But, it's usually a fallacy, since a logical argument ought to deal with reasons and justifications, and not try to undermine the other person's character. Ultimately, when we consider an argument, we should consider it on its own merits – that is, we should consider the reasons and justifications that are offered for the conclusion, and try to respond to those reasons directly. In the end, it shouldn't matter *who* is making the argument – even people with character flaws can produce valid forms of reasoning. So, attacking the *arguer* rather than the *argument* is not a rational form of argumentation.

3.9 Tu Quoque

Tu quoque is a Latin phrase which means, roughly, "You too!" or, "So do you!" Essentially, it involves trying to win an argument or avoid some conclusion by accusing the other person of hypocrisy. For instance:

(28) You're accusing me of cheating on the exam? I saw you copying the answers from Dave!

Notice that this is a fallacy because even if the other person *did* copy the answers from Dave, that doesn't make it any more or less likely that the speaker cheated on the exam as well. Tu quoque is actually a *kind* of ad hominem argument – it is attacking the other person's character, but specifically by calling them a hypocrite. But, just like with ad hominem arguments in general, even hypocrites can produce arguments that are, in themselves, persuasive. For example, suppose someone is arguing that smoking cigarettes is bad and dangerous, but they are addicted to smoking as well. That might make them somewhat of a hypocrite, but it doesn't mean that they are making a weak argument. So if you want to engage in a logical debate, then you must address the argument itself, and not the arguer.

3.10 Hasty Generalization

Hopefully, this fallacy should be pretty intuitive to grasp – we've probably all been guilty of this kind of reasoning at some point in the past. A hasty generalization is simply when you form a general conclusion about some group or class, but you do so on the basis of just a few examples. So, for instance:

(29) Alice is an adult and she's tall. Bob is an adult and he's also tall. So, I guess all adults are tall.

Obviously, one can't form a conclusion about all adults on the basis of just two examples. That would be jumping to a conclusion. We looked at examples like this when discussing inductive strength and weakness. Recall that an inductively strong argument is one in which the premises make the conclusion very *likely* to be true. So, *if* the premises are true, then the conclusion is *probably* true as well. Whereas, a weak argument is one in which even if the premises *are* true, that still wouldn't mean that the conclusion was all that likely. Clearly, this is an example of weak, rather than strong, inductive reasoning. Even if Alice and Bob *are* tall, that doesn't make it very probable that all adults are tall, since there are billions of adults. So, this is a *hasty generalization*, or a kind of *jumping to a conclusion*. It's clearly a fallacy because it's a poor way to reason. A good inductive argument should appeal to a strong body of evidence to support the conclusion. If you form a hasty generalization, then you're likely to be wrong much of the time. So, it's definitely something to avoid.

3.11 Weak Analogy

An argument exhibits a weak analogy when it tries to appeal to an analogy, or similarity, between two domains, say, X and Y, but in fact X and Y aren't really that similar, or the analogy is a poor one. For example:

(30) You shouldn't go to the concert. Going to a concert is like climbing a mountain – it's dangerous and cold.

This strikes me as a pretty strange argument. Is going to a concert really like climbing a mountain? Maybe it's true that some concerts are dangerous, but otherwise it doesn't seem like a very intuitive or powerful metaphor for going to a concert. Now, analogical thinking, thinking in terms of analogies, is a very powerful and important cognitive tool. So it's definitely appropriate to do so in certain circumstances. But the analogy must be a *good* one – one that really sheds light on the two things that are being compared. If you rely on a weak analogy, then it doesn't make sense to draw any inferences about one thing on the basis of the other, since they're not all that similar in the first place.

3.12 Begging the Question

Begging the question frequently comes up in philosophical debates. It refers to an argument which contains a premise that already assumes that the conclusion is true, or, in other words, the conclusion is kind of just a restatement of the premises. For example, consider this argument:

(31) Carol is lying about her grade. I know she is because she's not telling the truth.

Now, suppose you were trying to convince someone that Carol is lying about her grade. Normally, we assume that an argument is directed towards someone who doesn't already agree with your conclusion. So, let's suppose you're trying to argue that Carol is lying about her grade, and you're arguing with someone who thinks she's not lying. Would they find this argument convincing at all? The problem is, if you don't already believe that Carol is lying about her grade, then you're not going to believe that she's not telling the truth. The premise and the conclusion basically state the same thing! So, obviously that doesn't make for a very persuasive argument. A good argument is supposed to persuade someone who *doesn't* already accept the conclusion. And it's supposed to do so by using premises that the other person will accept and agree with. But if an argument begs the question, then it won't be very persuasive. If the premise and the conclusion are basically saying the same thing, then the other person won't be inclined to accept the premises if they don't accept the conclusion. So, basically, it's not likely to convince anyone.

It's worth mentioning that people sometimes use the phrase "begging the question" to really mean *raising the question.* So, for instance, someone might say: "We will soon be able to make trips to nearby planets. This begs the question – is space travel safe?" This doesn't really beg the question, at least in the philosophical sense. Rather the sentence might *raise* the question of whether space travel is safe, but it doesn't *beg the question.* In other words, it doesn't assume, as a premise in an argument, that space travel is safe. So, it's worth clarifying that begging and raising the question are two different things.

3.13 Equivocation

OK, next let's look at another kind of fallacy that is fairly widespread – equivocation. Equivocation is when an argument uses an ambiguous word in two different ways, but it ignores the ambiguity, and acts as though the word had one meaning. So, for instance:

(32) Justin Bieber is a star. Stars are giant astronomical objects. Therefore, Justin Bieber is a giant astronomical object.

Hopefully, the failure of reasoning in this argument should be clear to you. When the person says, "Justin Bieber is a star," then clearly they mean that Bieber is a famous celebrity, which is true. But when they say, "Stars are giant astronomical objects," then they're using a different meaning of 'star!' So it's true that stars, in the astronomical sense, *are* giant astronomical objects. And it's true, in the celebrity sense, that Justin Bieber is a star. But you can't connect those two statements on the basis of the word 'star,' since it has a different meaning in the two sentences! So, equivocation is when an argument ignores the fact that a certain word is ambiguous, and so draws faulty inferences on the basis of the assumption that it has a single meaning.

3.14 Appeal to Majority

An argument involves appeal to majority when it tries to argue that something is true on the basis of the fact that most people believe that it's true. For instance, let's go back to a time when people believed that the Earth was the center of the universe, and the Sun and stars revolved around the Earth. Then this guy Galileo gets up and says that the Earth actually revolves around the Sun, and someone responds by saying:

(33) The Sun revolves around the Earth. Come on, everyone knows that!

This is a fallacy because the fact that some belief is commonly held doesn't necessarily make it true. People can be wrong about all sorts of things, and sometimes lots of people can be wrong at the same time. So, you can't prove a point just by saying that other people agree with you. That's not going to be very convincing to someone who disagrees with you. And it's certainly not a valid form of reasoning, since human history contains all sorts of cases where basically everyone was wrong about a given question.

3.15 Appeal to Ignorance

This fallacy is a little bit more subtle. An argument involves an appeal to ignorance when it tries to form a conclusion on the basis of a *lack* of proof against the alternatives. For instance, suppose we're trying to debate who stole the cookies that were lying on the counter. Someone says:

(34) Alice stole the cookies. You've got no proof that anyone else did it!

This is a fallacy because the lack of evidence *against* your conclusion doesn't necessarily make it true. Maybe there's no proof at all about who stole the cookies – still, you can't leap to the conclusion that it was Alice just because we can't prove that it was Bob or Carol or Dave. The idea behind this fallacy is sometimes expressed in the phrase, "The absence of evidence is not evidence of absence." In other words, just because you can't find evidence of something doesn't necessarily mean that it's not the case. So, in the example we just looked at, the person is assuming that because there is no evidence that anyone else took the cookies, then it *wasn't* anyone else, so it must be Alice. This is a very weak form of reasoning though. You can't reliably infer that something is the case just because we haven't been able to prove the opposite.

3.16 Division

Now, we'll look at two related fallacies – division and composition. Both of these fallacies have to do with ignoring the differences between a whole and its parts. Let's start with division.

With division, a person infers that something is true about a thing's parts just because it is true about the thing as a whole. For example:

(35) Central Park is beautiful. Therefore, every single blade of grass in Central Park is beautiful.

Now, Central Park *is* beautiful, but that doesn't mean that every single part of Central Park, including every blade of grass, is also beautiful. You might think Central Park as a *whole* is beautiful, but not think that individual blades of grass are beautiful at all, even though blades of grass are part of what make up Central Park. So, in this case, it's a fallacy because you are taking something that is true about a thing as a whole, and you're assuming that it must also be true of all of its parts. But that's just not the case.

3.17 Composition

Composition is basically the opposite of division. With division, we infer *from* the whole *to* the parts. Whereas, with composition, we infer *from* the parts *to* the whole – we infer that something must be true about a thing as a whole on the basis of the fact that it's true about its parts. For example:

(36) The table is composed of atoms, and the atoms are moving really fast. Therefore, the table is moving really fast.

Once again, this is a poor form of reasoning. In general, what these examples show is that there's a difference between what's true about a thing and what's true about its parts. It's true that material objects are made of atoms that are moving very fast at a sub-atomic level. But just because it's true that the atoms are moving, we wouldn't say that the table is moving when it's just sitting there. So, once again, we see that what's true about the parts, isn't always true about the whole, and vice versa. When you make a fallacy by incorrectly inferring from something that's true about the parts (e.g. the parts are moving) to a conclusion about the whole (e.g., the whole is moving), then that's an example of the fallacy of composition.

3.18 Non Sequitur (Missing the Point)

In ordinary discourse, a non sequitur is a statement that comes out of the blue – it seems disconnected from the rest of the conversation. Similarly, in logic, a non sequitur is when a person draws a conclusion that isn't properly connected with the premises. So, for example:

(37) All dogs are furry, and Fido is a dog. Therefore, it's probably going to rain tomorrow.

Huh?

Where does the conclusion that it's probably going to rain tomorrow come from? It doesn't appear to be connected at all with the statements that came before it. It just comes out of the blue. So, clearly, this is a bad form of argument. An argument is supposed to exhibit a logical connection between the premises and the conclusion. Not all examples of non sequitur are this blatant, but it's obviously not a strong or valid way to reason.

3.19 Summary

In this chapter, we have examined a variety of logical fallacies that arise both in philosophical debate, and in everyday conversation. Having these concepts in your toolkit can make it easier to identify when someone is using one of these fallacies in their argument. It also helps clarify the distinction between good rational arguments and bad ones. Below is a list of the fallacies that we covered in this chapter. Although other fallacies exist, this is a fairly representative sample.

1. Appeal to authority
2. Appeal to force
3. Appeal to pity
4. Subjectivism
5. Straw man
6. Slippery slope
7. False alternative
8. Ad hominem
9. Tu quoque
10. Hasty generalization
11. Weak analogy
12. Begging the question
13. Equivocation
14. Appeal to majority
15. Appeal to ignorance
16. Division
17. Composition
18. Missing the point (non sequitur)

KEY CONCEPTS

- Logical fallacy

Part II

Symbolic Logic & Truth Tables

Chapter 1

Symbolic Logic Notation

1.1 Why use symbols?

Let's consider a few arguments:

Example (1)
(1) If Alice is late, the meeting will be delayed.
(2) Alice is late.
(3) Therefore, the meeting will be delayed.

Example (2)
(1) If Bob went to the supermarket, then he bought cookies.
(2) Bob went to the supermarket.
(3) Therefore, he bought cookies.

Example (3)
(1) If it rained this morning, the grass will be wet.
(2) It rained this morning.
(3) Therefore, the grass will be wet.

What do we notice about these arguments? Well, for one thing, they are all valid deductive arguments. Let's consider (1): The first premise states that the meeting will be delayed if Alice is late. Then the second premise affirms that Alice is, indeed, late. So, we can definitively conclude that the meeting will be delayed. We can observe that similar reasoning applies to the other arguments.

However, beyond the fact that they are all valid, these arguments clearly have a lot in common. They seem to follow a common pattern:

Pattern (1)
(1) If such-and-such, then so-and-so.
(2) Such-and-such.
(3) Therefore, so-and-so.

Let's dwell on this fact for a moment. Here, we are treating "such-and-such" and "so-and-so" as placeholders or *variables* for sentences. Thus, any argument will fit this pattern so long as each occurrence of "such-and-such" is replaced with the same sentence, and each occurrence of "so-and-so" is replaced with the same sentence, as in the following case:

Example (4)
(1) If Fido is a dog, then Fido is a canine.
(2) Fido is a dog.
(3) Therefore, Fido is a canine.

Is Argument 4 valid or invalid? Can we conceive of a counter-example? It doesn't seem to be possible for Fido to be a dog, but not a canine, given the information in premise (1). So, argument (4) is also a valid deductive argument. That might seem obvious, but wait a minute – the sentences "Fido is a dog" and "Fido is a canine" were totally arbitrary, but when we inserted them into the pattern described in Pattern 1, we ended up with a valid argument. So, that means we now know how to form a valid argument from any two sentences, just by following this pattern. That's a pretty significant fact.

Let's look at the pattern displayed in Pattern 1 more closely. This time, we'll use letters as variables for the sentences. That means that when you replace the variables with real sentences, you just have to make sure that *each occurrence of a given variable* is replaced with the *same sentence* as the other occurrences of that variable.

Pattern (1)
(1) If P, then Q.
(2) P.
(3) Therefore, Q.

We can now see that every instance of this pattern in which like variables are replaced with like sentences will result in a valid deductive argument.

Since validity only cares about the quality of reasoning involved in an argument, and ignores the question of whether the premises are true or false, this same generalization holds true even if we pick false statements as P and Q. For example, let's suppose P is "Fish are mammals," and Q is "Fish live in space":

Example (5)
(1) If fish are mammals, then fish live in space.
(2) Fish are mammals.
(3) Therefore, fish live in space.

Even though the conclusion is patently false, we've constructed another perfectly valid argument just by mechanically following Pattern 1.

Notice that Pattern 1 has almost entirely gotten rid of English words – the only remaining words are "If ..., then ..." and "Therefore." But "therefore" is really just a rhetorical device to clue you in to the conclusion — it is not essential. Thus, the only words that are needed to make a valid deductive argument (besides the sentences used to replace the variables) are "if" and "then."

Pattern (1)
(1) If P, then Q.
(2) P.
(3) Q.

Logicians are very interested in understanding these patterns. After all, we have seen that they are quite powerful – any specific argument that fits the pattern seems to be deductively valid. Thus, perhaps when judging whether an argument is valid or not, it can be helpful to see whether it fits into such a pattern of valid argument.

What we are beginning to look at, then, is the form of valid arguments. For, as we have seen, Pattern 1 remains valid *regardless of which sentences are used to replace "P" and "Q."* That is, you can use *any statement whatsoever* to replace a variable. So, since Pattern 1 produces a valid argument no matter which sentences are used to replace the variables, it must be in virtue of the *form* of the pattern itself that the argument is valid, and not in virtue of the specific claims that are used to replace, or *instantiate*, the variables. In other words, as long as it has the right form, the subject matter of the argument is totally irrelevant to whether or not it is valid. This is a highly significant fact: *validity depends on an argument's form, not its subject matter.*

Since we are really concerned with the form of the argument, then, we will do away with English words altogether, and think of "If ..., then ..." as an **operator** that takes two **sentences**. We will symbolize this function with the arrow, $\rightarrow$. Thus, the final form of Pattern 1 is:

Pattern (1)
(1) P
(2) $P \rightarrow Q$
___
(3) Q

"$P \rightarrow Q$" is read as: "If P, then Q;" or, equivalently, "P implies Q." We will symbolize the fact that (3) is intended to be the conclusion by putting it below a solid line.

We have now looked at one example of a deductive argument expressed in the notation of **symbolic logic**. The rest of the course will focus on symbolic logic (also known as formal logic). So, first we need to learn how to translate from English into symbolic logic.

First, let's look at one more easy example of an obviously valid argument form:

Example (6)
(1) Snow is white and grass is green.
___
(2) Therefore, snow is white.

Example (7)
(1) Alice is tall and she is wearing a blue hat.
___
(2) Therefore, Alice is tall.

Example (8)

(1) Bob is at home and he's watching TV.

(2) Therefore, Bob is at home.

These arguments are clearly valid. It is difficult to explain informally *why* they are valid, because the inference is so obvious and direct, but if it's true that both snow is white and grass is green, then obviously each statement must be individually true as well – i.e., snow is white. In general, if you say that two things are true, then they must be true individually as well.

These arguments are all clearly valid, and they also share a similar form:

Pattern (2)

(1) Such-and-such and so-and-so.

(2) Therefore, such-and-such.

Once again, we are informally using "such-and-such" and "so-and-so" as *sentential variables*. Let us, then, simply replace them with letters (and get rid of the unnecessary "therefore"):

Pattern (2)

(1) P and Q.

(2) P

We have now identified another valid sentence pattern, or form, where we can fill in the variables with *any sentences whatsoever* (as long as like variables are replaced by like sentences), and end up with a valid argument.

We can see that the only English word remaining is "and," so we will again think of this (informally for now) as a kind of operator that takes two arguments – P and Q, in this case – and produces a result (P). We will use the symbol & to represent "and," thus:

Pattern (2)

(1) $P \,\&\, Q$.

(2) P

This is another *valid inference rule* expressed in symbolic terms. In this unit, we will learn about a number of such rules, and how they can be used to determine whether an argument is valid or not.

1.2 Symbolic Logic Translations

1.2.1 Atomic vs. Complex Sentences

As we have already seen, when formalizing an English argument, that is, translating it into symbolic notation, we sometimes take *whole sentences* and replace them with variables. But with "If... then," we broke it up into two parts and used a different symbol for "If ..., then ...." In order to understand why, we must understand the concept of an **atomic sentence**.

First, notice that some sentences have other sentences as parts:

(9) The sky is blue.

(10) Grass is green.

(11) The sky is blue and grass is green.

Sentence (11) essentially consists of sentences (9) and (10) plus the word "and" – that is, it seems to have sentences (9) and (10) as parts. Whereas, it *doesn't* appear to be the case that either sentence (9) or sentence (10) has any other sentence as a part.

We will say, then, that a sentence which has another sentence as a part is a **complex sentence**, whereas a sentence that does not have any other sentences as parts is an **atomic sentence**. Sentences (9) and (10) are clearly atomic, while sentence (11) is complex.

1.2.2 Logical Operators

There are a number of special words in English that are used to combine sentences. Earlier we identified two such words (or sets of words): "and," and "if... then." We will think of "and" as an **operator** that combines two sentences – in other words, a **logical operator** (also known as a **sentential operator**, or, **logical connective**). A logical operator takes two sentences and produces a complex sentence.

Here is a list of logical operators in English:

Operator	Usage
AND	P AND Q
OR	P OR Q
IF ..., THEN ...	IF P THEN Q
IF	P IF Q
IF AND ONLY IF	P IF AND ONLY IF Q
UNLESS	P UNLESS Q

Notice that sometimes a logical operator will take two atomic sentences and produce a complex sentence, as in:

(11) The sky is blue and grass is green.

However, logical operators can also be used on complex sentences – to make even more complex sentences:

(12) If **the sky is blue** and **the grass is green**, then **this photograph will be beautiful.**

Note that sentence (12) actually contains *three* atomic sentences: (9), (10), and

(13) This photograph will be beautiful.

We can think of this as "and" combining "the sky is blue" and "the grass is green" to form "the sky is blue and the grass is green," and then "If. . . then" combines the already complex sentence "the sky is blue and the grass is green" and the atomic sentence "this photograph will be beautiful," to form the (doubly complex) sentence "**IF** the sky is blue **AND** the grass is green, **THEN** this photograph will be beautiful."

Finally, there is one logical operator that only operates on one argument: the word "not."

(14) Grass is green.

(15) Grass is not green.

Although the word "not" is located in the middle of the sentence, we can easily think of sentence (15) as consisting of sentence (14) plus the word "not." Thus, we will consider (15) a complex sentence as well, and we will take "not" to be a logical operator that only takes one argument.

1.2.3 Symbolic Notation

Here is a list of logical operators, and their accompanying translation into symbolic logic:

Operator	Usage	Symbolic Notation
AND (Conjunction)	P AND Q	$P \,\&\, Q$
OR (Disjunction)	P OR Q	$P \vee Q$
IF ... THEN ... (Conditional/Implication)	IF P THEN Q	$P \rightarrow Q$
IF AND ONLY IF (Biconditional)	P IF AND ONLY IF Q	$P \leftrightarrow Q$
NOT (Negation)	NOT P	$\sim P$

There are two other ways of expressing the conditional:

Operator	Usage	Symbolic Notation
IF	P IF Q	$Q \rightarrow P$
ONLY IF	P ONLY IF Q	$P \rightarrow Q$
UNLESS	P UNLESS Q	$\sim Q \rightarrow P$

1.2.4 A Note on "Unless" and the Oddity of Thought

"Unless" often throws students off: "P unless Q" gets translated as "$\sim Q \rightarrow P$". "Unless" encodes a negation (the "un" part), hence the negation in the formalism. When in doubt about how to translate think about it as you might try to understand a sentence. Children (and Pink Floyd fans) are often told that they can't have their pudding unless they eat their meat. That is:

$$\text{“No pudding unless meat”} = \sim \text{meat} \to \sim \text{pudding}.$$

Here there are two negations, one from unless, and the other from the explicit "No" in "no pudding."

Don't be fooled! You know what "unless" means, and have since you were a kid. Just think it through. Whenever you are stuck for a translation take a second to step back and think about how you would normally respond to what the sentence means. The sentence "A number is odd unless it is even," so "$\sim$ even $\to$ odd". In other words you know that "a number is odd unless it is even" just means "if it's not even then it's odd."

1.2.5 Other Uses of "Or" and "And"

So far, we have seen uses of "or"/"and" that clearly connect two sentences. But what about the following?

(16) Alice and Peter are tall.

(17) Bob is at home and is watching TV.

(18) Snow is white and cold.

These sentences all contain "and," but "and" doesn't seem to be combining two complete sentences. Consider sentence (18): in this case, "and" appears to be combining, or "operating on" – "Snow is white" and "cold." But "cold" is not a sentence. Nevertheless, if we had to rephrase what is conveyed by (18), we could easily do as follows:

(18*) Snow is white and snow is cold.

Obviously if snow is both white and cold, then snow is white, and snow is cold. So, the moral is that when you see "and" combining things that are not complete sentences, you should see if you can rephrase the sentence in a longer way so that both sides are complete sentences. Example:

(16*) Alice is tall and Peter is tall.

(17*) Bob is at home and Bob is watching TV.

The same lesson applies to "or," as we can observe from the following examples:

(19) Alice is at home or at the store.

(20) Either Bob or Claire is rich.

(21) Snow is white or heavy.

In these examples, "or" once again combines two parts that are not complete sentences. But we can easily rephrase these sentences so that they fit the familiar pattern for "or":

(19*) Alice is at home or Alice is at the store.

(20*) Bob is rich or Claire is rich.

(21*) Snow is white or snow is heavy.

So, if you encounter such examples of "and"/"or" that do not combine full sentences, try to reformulate them so that their logical structure is more transparent.

Hint

Be Careful! This strategy does not always work: "Alice and Bill drank a whole bottle of wine," does not mean the same as "Alice drank a whole bottle of wine and Bill drank a whole bottle of wine" – since they might have drank a single bottle *together*. We won't expand on this problem here, but it's something to be aware of.

1.2.6 Pragmatics

English communication does not simply involve conveying propositions in formal logic. Rather, language is full of connotations and nuances that we must learn to ignore in trying to analyze some English sentences.

For example, consider the following case:

(22) The sky is blue, but my shoes are untied.

How should we translate this sentence? We have not learned a symbol for the word "but." Should we then translate the whole sentence using one sentential variable? That seems inadequate, since (1) really does seem to contain sentential parts:

(23) The sky is blue.

(24) My shoes are untied.

So, "but" seems to be a sentential connective, or operator. But which one is it? Well, what can we infer from (22)? If (22) is true, then we can infer both (23) and (24). So, from a logical point of view, "but" seems to behave exactly like "and!" But isn't there a difference between saying "but" and saying "and"? Intuitively, "but" seems to imply some kind of contrast between the parts of the statement. That's, perhaps, why sentence (22) seems awkward -– what connection could there be between the sky being blue and the speaker's shoes being untied? Notice that this element of contrast is not always present when using "and":

(22*) The sky is blue, and my shoes are untied.

(22*) seems much more natural than (22), because "and" does not have that connotation of contrast. However, it is not the business of basic propositional logic to try to capture that difference in connotation.

What this example is supposed to illustrate is that there are many parts of an English sentence that are *not relevant* to our goal of translating into propositional logic. *Pragmatics* is the branch of the philosophy of language and linguistics that studies what is implied, as opposed to entailed, by sentences. *Entailment* is a logical property, but being merely implied isn't. For example, imagine you ask a scout about your prospects for being a professional basketball player and the scout responds "I think you'd make a great coach." There is nothing entailed in that statement about one's chances of being a professional basketball player. Nevertheless, pragmatically the statement implies that the scout doesn't think you will be a good player, for if he did he would've said so instead of changing the topic.

The study of pragmatics is of deep importance to understanding human communication, but it is outside of the bounds of what we will be considering in this course. Instead our focus is on the elements of the sentence that lead to the truth value of the sentence. Thus in the scout's sentence she is asserting that she thinks it is true that you would make a great coach, but she is not asserting anything about you as a player at all.

In order to understand the logical structure of an argument, when we analyze English into symbolic notation, we will simply ignore those parts of meaning that are not directly relevant to the *truth* of what is being asserted. Take sentence (25):

(25) The police arrived and the party stopped.

(25) Implies a certain causal and temporal structure: that the cops first arrived and as a consequence the party stopped. But nothing in the semantics of the sentence–nothing that determines the sentence's truth–is so implied. Instead the sentence is true as long as it's true that the police arrived and it's true that the party stopped. The causal and temporal implications are merely pragmatically implied–they aren't based on the logic of the sentence, but are instead based on extra-logical facts–facts about what one knows about the world (for example, that cops break up parties).

Going back to our original example about "but" and "and," "but" implies a contrast, but that contrast has no effect on the sentence's truth value - instead it just acts as a connective. Consequently, you may treat "but" as equivalent to "and."

1.2.7 Ambiguity

Groucho Marx once quipped:

(26) I once shot an elephant in my pajamas... How he got in my pajamas I'll never know.

When getting to the punchline, you realize that the first way you understood "I once shot an elephant in my pajamas"–with you shooting the elephant while wearing your pajamas–is not the only available reading. "I once shot an elephant in my pajamas" is *ambiguous* between meaning "While wearing pajamas, I shot an elephant" and "I shot an elephant who was wearing my pajamas."

Consider the following pairs of sentences:

(27) Alice is studying or Bob is studying, and Claire is studying.

(28) Alice is studying, or Bob is studying and Claire is studying.

Although the difference between the two sentences appears to be just a mere movement of a single comma, (27) and (28) suggest very different readings: (27) suggests that Claire is studying, *and* either Alice or Bob is also studying – this can only be true if Claire is indeed studying; (28), on the other hand, suggests that *either* Alice is studying, *or* Bob and Claire are studying – this suggests that Claire *might* be studying, but is not necessarily studying. These are clearly different conditions. But how are we to translate (27) and (28)? Even worse, suppose the commas were missing:

(27/28) Alice is studying or Bob is studying and Claire is studying.

Which reading ((27) or (28)) is the correct analysis of (27/28)? There is no clear answer. The problem is that, like ordinary speech, sentence (27/28) is ambiguous – in this case, it is *ambiguous* – its logical structure is not clear so the sentence is not *well-formed*.

This is a problem for doing logical analysis. For instance, how shall we analyze "or" in sentence (27/28)? Should "or" combine "Alice is studying" and "Bob is studying and Claire is studying"? Or should "or" simply combine "Alice is studying" and "Bob is studying"? Unfortunately, the sentence itself does not resolve the matter – it is logically ambiguous – but, our symbolic translations must *not* have any ambiguity. Thus, in analyzing a sentence like (3), we must resolve the ambiguity in one way or the other.

Typically, as in sentences (27) and (28), commas and other devices may help disambiguate the intended reading. In (27), the comma separates "Alice is studying or Bob is studying" and "Claire is studying." This suggests that "Alice is studying or Bob is studying" is a logical unit, and that "and" is operating on that unit and "Claire is studying."

In example (28), we see the opposite effect. The comma separates "Alice is studying" and "Bob is studying and Claire is studying." This suggests that the intended reading is that *either* Alice is studying, *or* Bob and Claire are studying.

In this latter case, we see that "Alice is studying" and "Bob is studying and Claire is studying" each form "units," in a way. Thus, we will symbolize this by enclosing such "units" in parentheses, to show that they go together.

Let us demonstrate how we would translate (27) and (28) using this method of disambiguation:

(27*) (Alice is studying or Bob is studying) and Claire is studying.

(28*) Alice is studying or (Bob is studying and Claire is studying).

As you can see, in the case of atomic sentences, like "Alice is studying," if it is not part of a bigger unit, we will simply drop the parentheses, as in (28*). This still results in a fully disambiguated representation.

Finally, let us translate these into symbolic logic using the following key:

P: Alice is studying	Q: Bob is studying	R: Claire is studying

Thus, we have:

(27) $(P \vee Q) \,\&\, R$

(28) $P \vee (Q \,\&\, R)$

Disambiguation is very important in symbolic logic because sentences like (27) and (28) can have different truth values, so *different disambiguations are not logically equivalent*. It is always crucial to check whether your logical formula is ambiguous or not.

Hint

A properly disambiguated complex sentence will have one pair of parentheses for every sentential operator minus one.

1.2.8 Semantic vs. Syntactic Ambiguity

Many English sentences are ambiguous. Some are ambiguities arise because of the meanings of words, and others because the structure of sentences. Take sentence (29)

(29) I dropped my money off at the bank.

On one reading - the natural one - one assumes bank refers to a financial institution and one dropped their money off there. But there is another reading available where "bank" refers to a river bank. (To make the context clear imagine you are about to take a dip in a river after a bank robbery and your accomplices ask you where you stashed the loot.) The sentence "I dropped my money off at the bank" is *semantically* ambiguous because one of the words it contains is ambiguous - "bank" can mean monetary bank or river bank.

Now let's return to the Marx quip which opened the section. When Groucho says "I once shot an elephant in my pajamas" he too is saying something ambiguous. But none of the words in "I once shot an elephant in my pajamas" are ambiguous themselves, instead the syntactic structure of the sentence is ambiguous. The sentence is *syntactically ambiguous* between two readings:

(30) [I once shot [an elephant] wearing my pajamas] (i.e., wearing my pajamas I once shot an elephant).

and

(30′) [I once shot an elephant] wearing my pajamas (i.e., I once shot an elephant who was wearing my pajamas).

Both types of ambiguities are disallowed in FOL, but they are dealt with differently. Syntactic ambiguities are to be disambiguated via parentheses. Semantic ambiguity on the other hand is dealt with by certain rules for how we specify keys. Take the Key we introduced above:

P: Alice is studying	Q: Bob is studying	R: Claire is studying

To exclude semantic ambiguities we must have certain rules for our keys. In particular each variable–here the P, Q, and R must have only one referent. That is, if "P" refers to "Alice is studying" then P CANNOT also refer to "Claire." The rule is that each variable only pick out one name. Call a variable that only refers to a single item a univocal variable. If we abide by this rule and then only build our sentences out of univocal variables and logical operators, then we can ensure our sentences will never be semantically ambiguous. As long as these rules are followed for one's key, then the only ambiguities to be on the lookout for are syntactic ambiguities.

But of course this supposes that the operators themselves are univocal. Thus, we must ensure that the definitions of the operators are themselves unambiguous. Return again to (25) (The police arrived and the party stopped.) In (25) we put the temporal and causal implication of "and" into the pragmatics of the sentence, and not into the meaning of "and." Instead "and" is only taken to be a mere function whose meaning is given by the following rules (where "and" is taken to be equivalent to &):

$$\begin{array}{l} P \,\&\, Q \\ P \\ \hline Q \end{array}$$

This says that from P and Q, one can infer P and one can infer Q. This rule of the meaning of "and" can be seen as an argument where $P \,\&\, Q$ make up the first premise, and there are two conclusions, the first being P, and the second being Q.

The meaning of "and" - the *conjunction* - is relatively straightforward. But other operators are a bit trickier. Let's now move to the trickiest, "or."

1.2.9 Exclusive vs. Inclusive "OR"

Students often have a problem with translating "or" as it appears in ordinary speech. In particular, sometimes "or" seems to imply that one or the other (or *both*) things might be true. For instance, consider the following utterance, by a speaker who is reluctant to go outside:

(31) It's going to be cold, or it's going to snow.

It seems like the speaker's statement is true if it's going to be cold, or if it's going to snow, or if it's both cold and it snows. Thus, it often seems that "or" is *inclusive* in the sense that it includes the possibility of both of its parts being true. Another example: Consider a hotel concierge discussing the benefits of the hotel:

(32) You can enjoy breakfast in bed or a nice swim in the pool.

The concierge is not implying that the guest can *only* do one or the other – both are *included*. Thus, we shall call this **inclusive "or"**.

On the other hand, other uses of "or" seem to suggest that only *one* argument *or* the other is true, but *not both*.

For instance: Imagine a banquet reception in which guests have a selection of main course – chicken or fish. The waiter utters:

(33) For the main course, you may have chicken or fish.

In uttering (33), the waiter seems to imply that the guest may have chicken *or* fish – *but not both*! (Only one entrée per guest.) Thus, this reading *excludes* the possibility of having *both chicken and fish*. Thus, we shall call it **exclusive "or"**.

The difference, then, between inclusive "or" and exclusive "or" is how they function when both of their component statements are true. With inclusive "or," which we can symbolize as V_I, the statement as a whole remains true even if *both* arguments are true. With exclusive "or" ($\vee_E$), the statement as a whole is *not* true when both arguments are true. This is a crucial difference.

Remember

Inclusive "Or" means *one-or-the-other-**or-both***. Exclusive "Or" means *one-or-the-other-but-**not-both***.

There are some reasons to suspect that the English use of "or" is the inclusive use. For one thing, note that it is never redundant to add the ***but-not-both*** in (33). This suggests that ***but-not-both*** cannot be part of the definition of "or" for if it was then adding it to "or" should make a redundancy. Compare: "bachelor" means "unmarried man" and calling someone an "unmarried bachelor" is redundant, as there are no unmarried bachelors. But hearing you can have *a-or-b-but-**not-both*** isn't at all redundant, it's informative and tells you about a restriction that wasn't apparent merely from the use of "or."

Thus we follow tradition in assuming that "or" always expresses inclusive-or, and thus we will understand "$\vee$" to mean "$\vee_I$." Thus for the purposes of symbolic logic translation, we will translate "or" as the inclusive disjunction ($\vee_I$).

1.2.10 Finding the Main Operator

Because sentences with multiple operators are disambiguated with parentheses, there is always one operator that is the **main operator** of the sentence. Finding the main operators are similar to finding the order of operations in arithmetic. $4 + 3 \times 7$ is taken to not be well-formed, and instead gets disambiguated as $4 + (3 \times 7)$. Here one first calculates the product of 3×7 and then adds four to that. Here addition is outside of the parentheses and so would get calculated last (as we will see in the next section). Similarly main operators are operators that take the greatest *scope* over the sentence - they hold over the rest of the sentence.

But that supposes there is always a main operator. In the case of sentences with no operators, there is no main operator:

(34) P

Main operator: NONE

In sentences with one operator, that operator must be the main operator.

(35) P if and only if Q

Main operator: "if and only if"

What about more complex sentences with more operators?

(36) P if and only if (Q or R)

What is the main operator in this sentence? As you can see, "or" does not operate on the whole sentence, so to speak – rather, it combines Q and R to make: Q or R. On the other hand, "if and only if" combines the parts of the whole sentence – (i) P, and (ii) (Q or R). In that sense, it seems like the *input* to "if and only if" is in fact the *output* of (Q or R). Therefore, there is always one "final" operator whose parts make up the whole rest of the sentence, and the output of this operator is not the input to any other operator. Another example:

(37) ((P and Q) or (Q and R)) if (P implies Q)

Here, the main operator is "if."

Hint

Another way to find the main operator, if the sentence is properly disambiguated, is to find the first operator that you encounter where there are *no open parentheses.*

KEY CONCEPTS

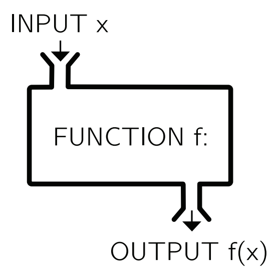
- Atomic sentence
- Complex sentence
- Logical/sentential operator
- Ambiguity (semantic vs. syntactic)
- Pragmatics

Chapter 2

Functions

We have seen that there are special words in English that are used to combine sentences – the logical, or sentential, operators. We have so far been talking loosely about “operators.” But what are these operators? Here, we will attempt to provide a somewhat more rigorous explanation.

In its most abstract, basic form, a function is just a mapping from inputs to outputs:



Think of a mathematical function, such as the square function.

$$X^2 = Y$$

Here X is the *input*, Y the *output*, and the function is to square the input in order to get an output. In the square function, the function takes an *input* – here an integer –, and produces an *output* – here another integer that is the square of the input. But a function needn’t take integers as inputs. It is intuitive to think of a function as a kind of “factory,” where inputs arrive at one end, are processed in some way inside, and then some output is produced. But one needn’t think of it as an integer factory–anything can, in certain circumstances, be an input into a function.

Strictly speaking, a mathematical function is a kind of *mapping* or *correspondence* between elements. For instance, you can think of the *square* function (over integers) as comprising the following mapping (with → standing in for “maps to”):

$$\begin{array}{lcl} \text{INPUT: 1} & \rightarrow & \text{OUTPUT: 1} \\ \text{INPUT: 2} & \rightarrow & \text{OUTPUT: 4} \\ \text{INPUT: 3} & \rightarrow & \text{OUTPUT: 9} \\ \text{INPUT: 4} & \rightarrow & \text{OUTPUT: 16} \\ \text{INPUT: 5} & \rightarrow & \text{OUTPUT: 25} \\ \vdots & & \vdots \end{array}$$

So, functions take *inputs* and map them to some *output* (which is typically the result of some computational function on the input).

Some functions are *one-to-one*, meaning that each input maps onto a unique output. The square function is one-to-one, as we can see: there are no two inputs that map onto the same output, and there is no input that maps onto more than one output. However, other functions are *many-to-one*, meaning that multiple inputs can result in the same output. Here is an example of a many-to-one function: *multiplication by 0*:

$$\begin{array}{lcl} \text{INPUT: 1} & \rightarrow & \text{OUTPUT: 0} \\ \text{INPUT: 2} & \rightarrow & \text{OUTPUT: 0} \\ \text{INPUT: 3} & \rightarrow & \text{OUTPUT: 0} \\ \vdots & & \vdots \end{array}$$

In this case, every input maps onto the same output, so this is a clear case of a many-to-one function. (Not all many-to-one functions have just one possible output.)

We can write this function as: $f(x) : x \times 0$

Here the $f(x)$ stands for "function of x", and x is a variable standing in for whatever is serving as the input. What comes after the colon defines the function, telling you what to do with the variable. Here it tells you to multiply the variable times 0.

Functions dealing with integers can have an infinity of outputs (consider the function: $f(x) : x + 1$. Yet when we consider *truth functions* (functions that take a proposition like a sentence as an input), there are really only two possible output values – `TRUE` and `FALSE`. There are an infinite number of propositions one can enter into a truth function (e.g., Michael Jordan was a basketball player, Michael Jordan was the greatest basketball player of all time, Michael Jordan was the greatest North Carolinian basketball player of all time, Michael Jordan was the greatest 6'6" basketball player of all-time, Michael Jordan was the greatest basketball player with a last name consisting of 6 letters, etc.). Since truth functions can take many possible inputs but can only have one of two outputs – `TRUE` or `FALSE` – truth functions are generally *many-to-one* — many different combinations of inputs could all lead to the same value, e.g., `TRUE`.

2.1 Basic Truth Functions and Truth Tables

It is useful to think of the *logical operators* or *connectives* that we have discussed previously as mathematical functions in this sense — that is, they map a set of inputs to a given output.

For instance, the `AND` function seems to take two *sentential arguments* and produces one sentence as its output. But, we have seen that the reason this is a valid rule is not in virtue of the *subject matter* of the sentences, but simply their *truth value*. Thus, whether or not a complex sentence formed with "and" will be true does not depend on the subject matter of the sentences, but simply on the truth values of the parts that it is operating on. In this sense, the output of the `AND` function is *completely determined* by the *truth values* of the sentential parts, and not their content.

Thus, logical connectives are *truth functions* because *they take sentences as inputs and compute the output (a truth value) strictly on the basis of the truth values of the inputs*.

Suppose, then, we take "and" to be a truth functional operator. How can we describe its behavior mathematically?

"And" can combine any infinite number of sentences, but since all it really "cares about" in the sentence is its truth value, there are really only four possibilities:

Take "P and Q" – If we just focus on the truth values of P and Q, we have the following possibilities:

P: `TRUE`	Q: `TRUE`
P: `TRUE`	Q: `FALSE`
P: `FALSE`	Q: `TRUE`
P: `FALSE`	Q: `FALSE`

Thus, we can fully describe the mathematical behavior of "and" if we just list the value for "P and Q" for each set of possible truth values. We will do so in the following table, known as a **truth table**:

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

The key insight is that this truth table tells you everything you need to know about the logical function of "and." Notice that it captures our intuition about the use of "and" as well. If someone says, "such-and-such and so-and-so," then their statement would be *false* if *either* "such-and-such" *or* "so-and-so" (or both) were false. This is captured in rows 2-4 of the table, where you can see that "P and Q" is false. As row 1 shows, "P and Q" is only true in the case where both P is true and Q is true.

Let us construct another truth table for the word "or."

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

This truth table once again reflects our ordinary intuitions about "or" – if someone says, "such-and-such, or so-and-so," then their statement will be false if *neither* such-and-such *nor* so-and-so is true. This is captured in row 4 of the table.

Let us now consider "If ..., then ...". The rules for "If... then ...', or what is known as the *conditional* or *implication*, are slightly more difficult than the rules for "and" and "or."

First, some brief terminology: "If ..., then ..." statements consist of two component parts — the part that goes after "if" and the part that goes after "then". The first part is called the **antecedent** of the conditional. The second part is called the **consequent** of the conditional. So, for a sentence like "If P, then Q", we shall say that P is the antecedent, and Q is the consequent.

Let us look at the truth table for "If ..., then ...":

P	Q	If P, then Q
T	T	T
T	F	F
F	T	?
F	F	?

Let us take an argument like:

(A) If it rained, the streets are wet.
(B) It rained.
(Z) Therefore, the streets are wet.

Notice that if it rained and the streets are wet (see row 1), then the statement "if it rained then the streets are wet" appears to be true. However, if it rained but the streets are *not* wet, then the statement "If it rained, then the streets are wet" appears to be incorrect, or false (see row 2). So far this is intuitive.

But what if it didn't rain? If it didn't rain at all, but the streets are wet, is statement (A) true or false? It's hard to say... (A) seems to be talking about what happens if it *did* rain – it doesn't really tell us what to do if it *didn't* rain.

It is hard to know what to say about "If P, then Q" when P is false. However, for the purposes of symbolic logic, it is standard to stipulate that when the antecedent of the conditional (P) is false, the whole conditional "If P, then Q" will be deemed *true* (no matter what the value of the consequent (Q) is.

So the truth table for "If... then" is:

P	Q	If P, then Q
T	T	T
T	F	F
F	T	T
F	F	T

Hint

This means that when evaluating a conditional, if you can tell that the antecedent is false, you can immediately tell that the whole conditional is true.

We will quickly introduce the truth tables for the remaining operators. We hope that by now you will be able to read a truth table and understand how it determines the logical function of a truth-functional operator.

The truth table for "if":

P	Q	P if Q
T	T	T
T	F	T
F	T	F
F	F	T

Notice that "P if Q" is logically equivalent to "If Q, then P". "Bob will come to the party if there is pizza there" says the same thing as "If there is pizza there, then Bob will come to the party".

The truth table for "only if":

P	Q	P only if Q
T	T	T
T	F	F
F	T	T
F	F	T

"P only if Q" is effectively the same as "If P, then Q".

The truth table for "if and only if":

P	Q	If P, then Q
T	T	T
T	F	F
F	T	F
F	F	T

"P if and only if Q" can be thought as the combination of "P if Q" (i.e., "If Q, then P"), and "P only if Q" (I.e., "If P, then Q"). This is known as the *biconditional*, because it essentially consists of the conjunction of two conditionals ($P \rightarrow Q$ and $Q \rightarrow P$). "P if and only if Q" is true just in case "P if Q" is true *and* "P only if Q" is true.

Another way to understand the biconditional is that the biconditional checks to see if both of its inputs have the *same truth value*, regardless of what that value is — if both the inputs are true, then the output is true; if one is true and the other false, the output is false; and, if both are false, then output is true.

Finally, let us look at the truth table for the *unary* operator (i.e., it takes only one argument) – "not":

P	Not P
T	F
F	T

Here, you can see that the whole function of "not" is to switch, or invert, the truth value of the input, so that it becomes its opposite.

2.2 More Complex Truth Tables

We have now seen how the truth tables for the basic logical operators look. It is important to try to memorize these tables. Although it may seem like a lot, these are the basis for analyzing propositions of any complexity. As we will see, when computing a truth table for a complex proposition, you simply apply the basic rules defined in the truth tables above over and over again. So, once you have mastered the basic truth table rules, you should be able to analyze a sentence of any complexity.

Let us take as an example the complex sentence "If (P and Q) then P":

P	Q	P and Q	If (P and Q) then P
T	T		
T	F		
F	T		
F	F		

As you can see, when constructing the truth table for the complex proposition "If (P and Q) then P," we have also included a column for the statement "P and Q," since this complex statement itself is one of the inputs to the "If ..., then ..." operator in the last column. But "P and Q" involves a basic truth table for conjunction, such as we have already seen. So, there should be no problem there. Let us fill that in now:

P	Q	P and Q	If (P and Q) then P
T	T	T	
T	F	F	
F	T	F	
F	F	F	

Now, how do we complete the last column? The key insight is that you can basically ignore the complexity of "(P and Q") when evaluating the last column. The arguments of the conditional in the last column are "P and Q" and "P" – but we have already computed the values for both of these (columns 1 and 3). So when we compute "If (P and Q) then P," we can simply refer to the columns that we already filled out (columns 1 and 3) to compute the value for "If (P and Q) then P."

P	Q	P and Q	If (P and Q) then P
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

In other words, at each column you are always computing a basic truth function whose arguments either have primitive truth values, or whose truth values have already been computed in a previous column. Therefore, to apply the next rule, you simply follow the pattern described in the basic truth table for that connective, using the appropriate previous columns in the table.

Since every truth function has at most two arguments, you will only ever have to look at *two* (!) other columns when computing the values for a given column. No matter how complex the final sentence is, the truth table can always be built up one operation at a time, so it only requires repeated application of the basic truth table rules.

Let's look at one more example:

P	Q	P or Q	Q if P	(P or Q) and (Q if P)
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

Here, we are trying to build the truth table for "(P or Q) and (Q if P)." First, we break it down by computing the columns for the embedded sentences "P or Q" and "Q if P." Then, we apply "and" to columns 3 and 4 to get the conjunction: "(P or Q) and (Q if P)." To do this, we just look at the columns for "P or Q" and "Q if P" and find the rows where they both say true.

By now, you should get the sense that this process is fully general – any complex proposition can be broken down into component parts, and each part is connected by a simple logical operator. Once you understand the truth tables for the logical operators, computing the truth table for a complex proposition simply consists in applying those rules to columns where the component propositions get more and complex as you proceed. In other words, you "build up" to the truth table of the final complex sentence by analyzing the truth values of the parts and combining them again and again.

KEY CONCEPTS

- Truth function
- Truth table

Chapter 3

Truth Tables and Validity

3.1 The Truth Table Method for Determining Validity

Being able to construct a truth table for complex propositions provides an important method that allows us to determine definitively whether an argument is valid.

Let us consider the following argument:

(1) P and Q
(2) If P, then R
(3) Therefore, R

Intuitively, this argument is valid. How can we show this using a truth table?

First, let's construct a complete truth table for every proposition in the argument, and combine them into one:

P	Q	R	P and Q	If P, then R
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Now, how do we use this table to determine whether the argument is valid? Well, recall that an argument is valid just in case if the premises are true, then the conclusion is true. That means we want to look for any row in the table in which *all the premises are true* (and we can ignore any other rows). You can see the relevant cells colored:

P	Q	R	P and Q	If P, then R	R
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	F	T	T
F	F	F	F	T	F

Row 1 is the only row where all the premises ((1) and (2)) are true. Now, in order to determine validity, you first find the rows where all the premises are true, then you see whether the *conclusion is true in all of those rows as well.* In this case, there is only one relevant row, and the conclusion (R) is indeed true in that row, so therefore the argument is valid. (Note that R was duplicated on the right to make the reading more intuitive, but this is, strictly speaking, not necessary.)

Let's look at another example:

(1) P and Q
(2) (P and Q) implies Q

The completed truth table for this argument looks like this:

P	Q	P and Q	(P and Q) implies Q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

As you can see, in this example "P and Q" is the only premise, so row 1 is the only row where all the premises are true; now, if we look to the conclusion at these rows, we see that it is true in that row. Therefore, it is clear that whenever the premise(s) are true, the conclusion is true, and hence this argument is valid.

Let us also notice something interesting about this last example: Consider the final column, which states the conclusion. Notice that it has `TRUE` in every entry.

P	Q	P and Q	(P and Q) implies Q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

This is in fact an example of a very significant class of propositions known as **tautologies**. Intuitively, a tautology is a statement that is true "no matter what," i.e., no matter what is "going on in the world." For example, "It's raining or it's not raining" might be considered a tautology, for it's always true that wherever you are, it either *is raining* or *it's not raining.* More precisely, a tautology is a statement whose column in the truth table contains all `TRUE` values.

In contrast to a tautology, there are statements that are false "no matter what" – they have `FALSE` in every row of their truth table column. For example: "Alice is at home and she's not at home." This is totally *impossible* or *contradictory* because it says two things that are totally incompatible. This is known as a **contradiction**. A contradiction is a statement that asserts an impossibility. It has all `FALSE` values in its truth table column.

Finally, some statements are true in some cases, and false in others. For instance, consider "If P then R" from Table 1 – this statement has a mixture of `TRUE` and `FALSE` values in its column. Intuitively, this suggests that the statement is *sometimes* true and *sometimes* false. This is known as a **contingent** statement.

These are importantly terminology to bear in mind as we move forward. Let us take stock:

Validity

An argument is **valid**, as shown in a truth table, just in case for every row in which every premise is `TRUE`, the conclusion is `TRUE` at that row as well. (All other rows can be ignored.)

Tautology

A statement is a **tautology** if the complete truth table shows that its value is `TRUE` at every row of the truth table. (Intuitively, it is a statement that is "always" true "no matter what the circumstances.")

Contradiction

A statement is a **contradiction** if the complete truth table shows that its value is `FALSE` at every row of the truth table. (Intuitively, it is a statement that is "always" false "no matter what the circumstances.")

Contingency

A statement is **contingent** (or, is a **contingency**) if the complete truth table shows that its value is `TRUE` at some rows and `FALSE` at others. (Intuitively, it is a statement that is whose truth value may change depending on the circumstances.)

KEY CONCEPTS

- Tautology
- Contradiction
- Contingent statement

Part III

Natural Deduction

Chapter 1

What is Natural Deduction?

In the last unit, we learned how to construct truth tables for symbolic logic propositions, and then we learned how to use these truth tables to judge whether or not an argument is valid. The truth table method is certainly effective, but it doesn't really reflect how people *actually* reason about deductive arguments. Essentially, the truth table method judges whether an argument is valid by looking at every possible way the world could be, and then seeing whether the conclusion is true in every possible state where the premises are true. But when we actually consider philosophical arguments, we usually don't keep track of every possible state of the world. Our brains simply can't handle that much information at once. Similarly, what would happen if we tried to build a truth table for the following argument?

Argument (1)

(1) p
(2) q
―――――――――――
(3) $(p \,\&\, q) \vee (r \,\&\, s)$

You can see the answer on the top of the next page.

To build a truth table for a proposition with two different variables, you only need four rows. To build one for a proposition with three variables, you need eight rows. And to build one for a proposition with four variables, you need sixteen! As you can see, the number of rows in a truth table is equal to 2 raised to the number of distinct propositional variables. That means that truth tables grow exponentially as you increase the number of distinct variables. So, this clearly isn't going to work for very complex arguments.

So, although the truth table method is effective for simple arguments, or for analyzing propositions with only a few variables, (i) it doesn't reflect the way human beings actually reason about logical arguments, and (ii) truth tables become too large and unwieldy if you have more than a few variables. In this

p	q	r	s	$p \& q$	$r \& s$	$(p \& q) \vee (r \& s)$
TRUE	TRUE	TRUE	TRUE			
TRUE	TRUE	TRUE	FALSE			
TRUE	TRUE	FALSE	TRUE			
TRUE	TRUE	FALSE	FALSE			
TRUE	FALSE	TRUE	TRUE			
TRUE	FALSE	TRUE	FALSE			
TRUE	FALSE	FALSE	TRUE			
TRUE	FALSE	FALSE	FALSE			
FALSE	TRUE	TRUE	TRUE			
FALSE	TRUE	TRUE	FALSE			
FALSE	TRUE	FALSE	TRUE			
FALSE	TRUE	FALSE	FALSE			
FALSE	FALSE	TRUE	TRUE			
FALSE	FALSE	TRUE	FALSE			
FALSE	FALSE	FALSE	TRUE			
FALSE	FALSE	FALSE	FALSE			

Truth table for Example (1)

unit, we'll learn about a proof method known as **natural deduction**, which doesn't have these problems.

1.1 A More Natural Kind of Proof

Rather than abstractly considering every possible state of the world, when we think about logical arguments, a much more *natural* approach is to consider the premises and try to draw out their obvious consequences. When we want to construct a large, complex proof, we will proceed step by step, building a chain of reasoning that starts with the premises and ends with the conclusion. This is much closer to how natural deduction works.

Natural deduction works by identifying basic patterns of good reasoning. One way to think about it is like a game. We will identify a set of basic rules, which are inference patterns that we know for sure are valid forms of reasoning. Then, the goal is to see whether you can move from the premises to the conclusion by applying some combination of these rules. If you can, then you have shown that the argument is valid.

Let's start by looking at an example of one of the simplest inference rules that we can imagine. Consider the following arguments:

Argument (2)

(1) Alice is tall and Bob is short.

(2) Therefore, Alice is tall.

Argument (3)

(1)	Carol is late and Dave is early.
(2)	Therefore, Carol is late.

Argument (4)

(1)	Alice is not home and Bob is at work.
(2)	Therefore, Alice is not home.

Argument (5)

(1)	It's raining and it's cold out.
(2)	Therefore, it's raining.

As you can see, these arguments all have basically the same form. The first statement, the premise, consists of a conjunction between two statements (*p and q*). Then, the conclusion asserts that one of those statements, or conjuncts, is true (p):

Argument (6)

(1)	$p \& q$
(2)	p

Is this a valid form of reasoning or not? Well, let's assume the premise is true. If a conjunction, like *p and q*, is true, then each of the component statements, or conjuncts, (p, and q) must also be true individually. So, whenever you have a conjunction, you know that you can infer from the conjunction either of the conjuncts. In these examples, it is the first conjunct that is the conclusion, but you could just as easily infer the second conjunct:

Argument (7)

(1)	Alice is tall and Bob is short.
(2)	Therefore, Bob is short.

Argument (8)

(1)	Carol is late and Dave is early.
(2)	Therefore, Dave is early.

Argument (9)

(1)	Alice is not home and Bob is at work.
(2)	Therefore, Bob is at work.

Argument (10)

(1)	It's raining and it's cold out.
(2)	Therefore, it's cold out

Argument (11)

(1)	$p \& q$
(2)	q

So, we have identified one very basic pattern of valid reasoning: Whenever you have a conjunction in your proof, you can infer that each of the conjuncts is true individually. In other words, the move from $p \& q$ to p, or from $p \& q$ to q, is a "legal" move in the game.

Let's see how we would use this rule to construct a proof. Suppose we wanted to construct a proof for the following argument:

Argument (12)

(1)	$p \,\&\, (q \,\&\, (r \,\&\, s))$
(2)	s

According to the rule that we've just introduced, whenever we have a conjunction, we can infer each of the conjuncts. Well, our premise (1) just happens to be a conjunction. So, what can we "do" with it? Well, we can add each of the conjuncts (p, the left conjunct, and $q \,\&\, (r \,\&\, s)$, the right conjunct), to our proof.

(1)	$p \,\&\, (q \,\&\, (r \,\&\, s))$
(2)	p
(3)	$q \,\&\, (r \,\&\, s)$
(4)	s

Here, we take the information in premise (1), which is a conjunction, and "break it apart" and draw out each of the conjuncts individually. So, if $p \,\&\, (q \,\&\, (r \,\&\, s))$ is true, then we can infer p and also $q \,\&\, (r \,\&\, s)$.

Great! But, we still haven't reached our conclusion - s. So, what can we do? Well, we only have one rule so far, so let's apply it again. This time, we'll apply the same rule to the sentence that we just added to the proof (3), so we get:

(1)	$p \,\&\, (q \,\&\, (r \,\&\, s))$
(2)	p
(3)	$q \,\&\, (r \,\&\, s)$
(4)	q
(5)	$r \,\&\, s$
(6)	s

Here I've applied the conjunction rule (twice, actually) to the statement $q \,\&\, (r \,\&\, s)$, first to infer q (line 4), then to infer $r \,\&\, s$ (line 5). So, as you can see, you can apply the rules again and again until you eventually reach your goal. We seem to be a little closer now, but we still only have $r \,\&\, s$, and we're trying to prove s. So, we just need to apply the rule one more time to $r \,\&\, s$:

(1)	$p \,\&\, (q \,\&\, (r \,\&\, s))$
(2)	p
(3)	$q \,\&\, (r \,\&\, s)$
(4)	q
(5)	$r \,\&\, s$
(6)	r
(7)	s

Here, I've applied the very same rule, this time to the statement $r \,\&\, s$, to infer both r and s. Since s is the statement we were trying to prove, we're done! We've shown that you can move from $p \,\&\, (q \,\&\, (r \,\&\, s))$ to s by repeatedly applying a very basic rule of inference that we know is definitely valid. Since we've shown that you can create a valid chain of reasoning that starts with the premises and ends with the conclusion, and only performs basic inferences that are clearly valid, we've proven that you can infer s from $p \,\&\, (q \,\&\, (r \,\&\, s))$, i.e., that argument (12) is valid.

That's the basic idea behind natural deduction proofs. Here, we've constructed a proof using just one inference rule. In the rest of the unit, we'll learn a bunch of other simple rules, but the basic strategy will be just like what we've done here.

Notice that unlike the truth table method, which is entirely mechanical, natural deduction proofs are *constructive* proofs - they require some creative input on your part. There is no automatic mechanism to tell you *how* to get from the premises to the conclusion, or what sequence of rules you need to apply, so it's a bit more like solving a puzzle that requires creative problem-solving.

1.2 Notation For Natural Deduction Proofs

It's important to use the right notation when constructing natural deduction proofs. If the goal is to construct a valid chain of reasoning from the premises to the conclusion, you need to be very clear about how you are getting from one line to the next. So, in addition to writing the statements down in order, like we did in the previous section, we will also want to make note of (i) which rule is being applied, and (ii) which statement(s) it is being applied to.

In order to make this easier, each of the basic inference rules will be given a name, so that when we apply that rule, we can refer to it in the notation. Consider the conjunction rule that we have been applying:

$$\begin{array}{ll} (1) & p \,\&\, q \\ \hline (2) & p \end{array}$$

In formal notation, this rule is known as **Simplification** – essentially, you can think of it as taking a complex statement (a conjunction) and *simplifying* it (reducing it to one of its conjuncts).

Simplification

Left rule: $\begin{array}{ll} (1) & P \,\&\, Q \\ \hline (2) & P \end{array}$ Right rule: $\begin{array}{ll} (1) & P \,\&\, Q \\ \hline (2) & Q \end{array}$

(Although these are *technically* distinct "moves," we will refer to them both as "Simplification," i.e., it doesn't matter if you are deriving the first conjunct or the second.)

Now, when we are building our proof, we will refer to this rule each time it is applied. The following is a complete proof from the premise $p \,\&\, (q \,\&\, (r \,\&\, s))$ to the conclusion s:

(#)	*Proposition*	*Justification*
(1)	$p \,\&\, (q \,\&\, (r \,\&\, s))$	Premise
(2)	p	Simplification (1)
(3)	$q \,\&\, (r \,\&\, s)$	Simplification (1)
(4)	q	Simplification (3)
(5)	$r \,\&\, s$	Simplification (3)
(6)	r	Simplification (5)
(7)	s	Simplification (5)

So, a proof has two columns - on the left-hand side, you write the statements that make up the proof, and on the right-hand side you write down the inference rule that you used to derive that sentence, and which previous sentence(s) in the proof you are referring to when you use the rule. (If the statement is actually one of the *premises* of the argument, you can simply write "Premise.") This allows the reader to follow the chain of reasoning and make sure that each move is a valid one. For instance, the proof above shows that line 5 ($r \,\&\, s$) was derived from line 3 ($q \,\&\, (r \,\&\, s)$) using the Simplification rule.

It's worth noting that this proof is a little bit inefficient - some of the inferences that are made don't really help me get to the conclusion. For instance, on line 2, I infer p from $p \,\&\, (q \,\&\, (r \,\&\, s))$. But I don't actually "do" anything with p after that line; I don't use the information in line 2 anywhere else in the proof (as you can easily tell with our notation, since no line later on refers to it). That means we could have done without it. In general, with natural deduction, you will often be able to apply the rules in ways that are "legal" but don't really help you get to your conclusion. This can definitely be helpful when exploring solutions, and it doesn't actually result in an *incorrect* proof, but it's best to leave those kinds of moves out when we are constructing proofs. So, here is the same proof again, but with only the "essential" lines:

(#)	*Proposition*	*Justification*
(1)	$p \,\&\, (q \,\&\, (r \,\&\, s))$	Premise
(2)	$q \,\&\, (r \,\&\, s)$	Simplification (1)
(3)	$r \,\&\, s$	Simplification (2)
(4)	s	Simplification (3)

Now, it's nice and clear that we are repeatedly simplifying the complex conjunction in the premise (1) until we reach the conclusion, and each step actually brings us closer to the goal. Notice, of course, that you can only apply Simplification once per line. E.g., I can't go from line 1 to line 3 directly using one application of Simplification (1), because that would be skipping a move.

So, that's the basic idea behind natural deduction. The real trick is learning all of the rules that are available to us, and coming up with strategies that make it easier to construct correct proofs.

KEY CONCEPTS

- Natural deduction
- Natural deduction rules

Chapter 2

Rules for Natural Deduction: Rules of Implication

In this chapter, we will introduce some of the basic rules for natural deduction proofs.

2.1 Simplification

Simplification

Left rule:			Right rule:		
	(1)	$P \mathbin{\&} Q$		(3)	$P \mathbin{\&} Q$
	(2)	P		(4)	Q

Example argument

(1)	$p \mathbin{\&} (q \mathbin{\&} (r \mathbin{\&} s))$
(2)	q

Proof

(1)	$p \mathbin{\&} (q \mathbin{\&} (r \mathbin{\&} s))$	Premise
(2)	$q \mathbin{\&} (r \mathbin{\&} s)$	Simplification (1)
(3)	q	Simplification (2)

As we've already discussed, simplification is just a matter of reducing a conjunction to one of its conjuncts. Since a true conjunction requires each one of the conjuncts, or component statements, to be true, whenever you have a conjunction in your proof, you know that each side of the conjunction must be true individually.

2.2 Conjunction

Conjunction	
(1)	P
(2)	Q
(3)	$P \& Q$

Conjunction is kind of the opposite of Simplification. The most basic fact about a conjunction is that a conjunction is true whenever each of its conjuncts are true. So, the rule of Conjunction states that if you have a proof that asserts two statements, e.g., p, and q, on separate lines, then you can infer the conjunction of those statements as well, i.e., $p \& q$.

Note that whereas Simplification is a rule that only needs to cite one previous line, Conjunction involves "gluing" together two individual statements into one conjunction – so, it requires you to refer to two previous lines in the proof.

Example argument

(1)	$p \& q$
(2)	r
(3)	$p \& r$

Proof

(1)	$p \& q$	Premise
(2)	r	Premise
(3)	p	Simplification (1)
(4)	$p \& r$	Conjunction (2,3)

The proof above uses both Simplification and Conjunction to reach the conclusion. First, Simplification is applied to line 1, and we extract the left-hand conjunct. Then, since we now have both p and r asserted independently in the proof (lines 2 and 3), we can combine them into $p \& r$ using Conjunction (line 4).

2.3 Modus Ponens

Conjunction	
(1)	P
(2)	$P \rightarrow Q$
(3)	Q

Modus Ponens is one of the most important rules of inference, since it captures the essence of the conditional, or implication. What do we know about a conditional? If a conditional is true, like $p \rightarrow q$, we still don't know whether p is true or whether q is true. But since it's a conditional, we know that *if* p is true, *then* q is true as well. So, with Modus Ponens, if we have a conditional somewhere in our proof, and we *also* have asserted the *antecedent* (left-hand side) of the conditional independently, then we know that the *consequent* (right-hand side) of the conditional must be true as well. If the conditional is true, then one thing we know *isn't* the case is that the antecedent is true and the consequent false. So, once we have asserted the antecedent, we can go ahead and assert the consequent as well.

Example argument

(1)	$p \to (q \,\&\, r)$
(2)	p
(3)	r

Proof

(1)	$p \to (q \,\&\, r)$	Premise
(2)	p	Premise
(3)	$q \,\&\, r$	Modus Ponens (1,2)
(4)	r	Simplification (3)

In this proof, we use Modus Ponens to combine the conditional (line 1), and the antecedent of the conditional (line 2), and we derive the consequent (line 3). Then we apply Simplification to 3 in order to derive the right-hand conjunct (line 4).

2.4 Modus Tollens

Modus Tollens

(1)	$P \to Q$
(2)	$\sim Q$
(3)	$\sim P$

Modus Tollens is similar in some respects to Modus Ponens - you can think of it as kind of the inverse of Modus Ponens. It also builds off of the basic logical properties of the conditional. As you'll recall, the one thing we know about a conditional is that *if* the antecedent is true, *then* the consequent is true as well. So, that means if the consequent is *false*, then the antecedent *can't* be true either! If you think about it in terms of the truth table, the only way a conditional can be true when its consequent is false, is if the antecedent is false as well (since the conditional is always true when the antecedent is false, no matter what).

So, whereas with Modus Ponens, we have a conditional and its antecedent, and we derive the consequent, with Modus Tollens, we have a conditional and the *negation* of the *consequent*, and we derive the *negation* of the *antecedent*.

It both reverses the order of arguments compared to Modus Ponens and also "flips the sign" of the arguments.

Think about it in terms of an intuitive example:

(1) If Dave is home, then the lights are on.
(2) The lights are not on.
(3) Therefore, Dave is not home.

The fact that the lights are not on demonstrates that Dave couldn't be home, because if he *were* home, then the lights would be on (following line 1). So we go from the negation of the consequent (line 2) to the negation of the antecedent (line 3).

Example Argument

(1)	$p \,\&\, (q \to {\sim}p)$
(2)	${\sim}q$

Proof

(1)	$p \,\&\, (q \to {\sim}p)$	Premise
(2)	p	Simplification (1)
(3)	$q \to {\sim}p$	Simplification (2)
(4)	${\sim}q$	Modus Tollens (2, 3)

Here, we use Simplification to split up the conjunction (line 1) into independent statements (lines 2 and 3). Now, we have a conditional (line 3), *and* we have the *negation* of the *consequent* (line 2, since p is the negation of ${\sim}p$). That means we can use Modus Tollens to derive the *negation* of the *antecedent* (line 4).

2.5 Addition

Addition

Left rule:			Right rule:		
	(1)	P		(3)	P
	(2)	$Q \lor P$		(4)	$P \lor Q$

Recall that for a disjunction to be a true, only *one* of the disjuncts (component statements) has to be true. So, if we have a proof where we've already asserted some statement, like p (although it could also be a complex statement like $p \to (p \,\&\, r)$), then you can always add a *disjunction* with that statement as one of the disjuncts, and *any other statement* as the other disjunct. In other words, if you have a true statement, you can add *any statement whatsoever* to it using disjunction (hence the name 'Addition'). Since one of the disjuncts is sure to be true, it really doesn't matter whether the other disjunct is true or false, so it can be whatever you like. As in:

(1)	$p \to q$
(2)	$(p \to q) \vee (r \,\&\, s)$

(1)	$p \leftrightarrow (p \vee s)$
(2)	$(p \leftrightarrow (p \vee s)) \vee (r \to p)$

(1)	$p \,\&\, (q \,\&\, s)$
(2)	$(\sim r \vee p) \vee (p \,\&\, (q \,\&\, s))$

In each of these cases the premise, or statement above the line, is used as a disjunct in a more complex disjunctive statement as the conclusion (below the line). As you can see, this rule of Addition can work on simple statements or complex statements. And, because the order of arguments in a disjunction doesn't matter at all for its truth value, you can take the statement you are adding to the disjunction and put it on the left-hand side or the right-hand side.

Example argument

(1)	$p \to q$
(2)	p
(3)	$q \vee (r \to s)$

Proof

(1)	$p \to q$	Premise
(2)	p	Premise
(3)	q	Modus Ponens (1, 2)
(4)	$q \vee (r \to s)$	Addition (3)

Hint

Addition is a useful rule to apply if the conclusion of the argument doesn't appear anywhere in the premises. E.g., $r \to s$ doesn't appear anywhere until the last line of the proof. So the only way you could really derive it would be with addition.

2.6 A More Complex Example

Let's look at a proof that uses all of the rules that we've seen so far.

Example Argument

(1)	$q \,\&\, ((q \vee r) \to \sim s)$
(2)	$p \to s$
(3)	$\sim s \,\&\, \sim p$

Proof

(1)	$q \,\&\, ((q \vee r) \to {\sim}s)$	Premise
(2)	$p \to s$	Premise
(3)	q	Simplification (1)
(4)	$(q \vee r) \to {\sim}s$	Simplification (1)
(5)	$q \vee r$	Addition (3)
(6)	${\sim}s$	Modus Ponens (4, 5)
(7)	${\sim}p$	Modus Tollens (2, 6)
(8)	${\sim}s \,\&\, {\sim}p$	Conjunction (6, 7)

Let's walk through this proof. Lines 1 and 2 are simply the premises. Then, we use Simplification on the conjunction in line 1 to derive each of its conjuncts in lines 3 and 4. Now, notice that 4 is a conditional, and the antecedent is $q \vee r$. We don't have that statement yet, but we *do* have q, and because of the rule of Addition, you can create a disjunction out of any true statement. So, we have q, and we use Addition to "add" r to make $q \vee r$.

Hint

Be careful! The rule of addition does NOT work for *conjunctions*, only *disjunctions*!

Then, we have derived the antecedent for the conditional in like 4, so we can use Modus Ponens to infer the consequent (line 6). So how do we get to ${\sim}p$? Well, line 2 is also a conditional, and remember that Modus *Tollens* tells us that whenever we have a conditional, and we have the *negation* of the *consequent*, we can infer the negation of the antecedent. So, we have Modus Tollens on lines 2 and 6 to derive ${\sim}p$ (line 7). Finally, our last two lines of the proof are now ${\sim}s$ and ${\sim}p$, and according to Conjunction, we can create a conjunctive statement out of any two individual lines. So, we just combine lines 6 and 7 to derive the ultimate conclusion of the argument, line 8, ${\sim}s \,\&\, {\sim}p$. And we're done!

2.7 Strategies

- If you find a conjunction, always consider using Simplification to separate out the individual conjuncts. It might be helpful, and it can never really hurt.

- If you see a conditional, try to think about how you can derive the antecedent of the conditional. Usually, you'll want to use Modus Ponens or Modus Tollens for a conditional, so consider the other premises and see if you can't convert them into either the antecedent of the conditional or the negation of the consequent.

- If you have a statement or a propositional variable that seems to "come out of nowhere" and it's part of a disjunction, then try to prove one of the disjuncts, and then use Addition to add the new information to it.

- Conjunction is perhaps not used as often as some other rules, but always keep in mind that sometimes you may need to create a conjunction – say, you want to do something with the statement $(p \,\&\, q) \rightarrow r$, then see if it's possible to establish each of the conjuncts p, and q, independently, so that you can then combine them with Conjunction.

2.8 Hypothetical Syllogism

Hypothetical Syllogism	
(1)	$P \rightarrow Q$
(2)	$Q \rightarrow R$
(3)	$P \rightarrow R$

You can thinking of hypothetical syllogism as "chaining together" two conditional statements. If we have two conditionals in our proof where the *consequent* of the first is exactly the same as the *antecedent* of the second, then you can infer a new conditional that has the *antecedent* of the first and the *consequent* of the second. In other words, if there is a "common link" between the consequent of one conditional and the antecedent of another, then you can essentially create a "shortcut" that bypasses that common link. For instance, consider:

(1)	$p \rightarrow (q \,\&\, r)$	Premise
(2)	$(q \,\&\, r) \rightarrow s$	Premise
(3)	$p \rightarrow s$	Hypothetical Syllogism (1, 2)

In this example, $q \,\&\, r$ is a common link between the conditional in line 1 and line 2. So we can directly "hop over" $q \,\&\, r$ and connect p and s with a conditional. To put it in metaphorical terms, line 1 is like a train that takes you from p to $q \,\&\, r$; line 2 is like a train that takes you from $q \,\&\, r$ to s. So, that means that we can be sure that there is a path to take us from p to s, and we can simply leave out the intermediate "stop" of $q \,\&\, r$.

Example Argument

(1)	$p \rightarrow r$
(2)	$\sim r \vee s$
(3)	p
(4)	$s \vee t$

Proof

(1)	$p \to r$	Premise
(2)	$\sim r \lor s$	Premise
(3)	p	Premise
(4)	$r \to s$	Material Equivalence (2)
(5)	$p \to s$	Hypothetical Syllogism (1, 4)
(6)	s	Modus Ponens (3, 5)
(7)	$s \lor t$	Addition (6)

2.9 Constructive Dilemma

Constructive Dilemma	
(1)	$P \to R$
(2)	$Q \to S$
(3)	$P \lor Q$
(4)	$R \lor S$

This rule might seem a little more complicated at first, but it should be rather intuitive once you grasp it. The key is to think of the disjunction $p \lor q$ as saying, "Either p is the case, or q is the case." Then $p \to r$ tells us that "If p is the case, then r is the case," and $q \to s$ tells us that "If q is the case, then s is the case." So, since $p \lor q$ is a disjunction, we don't know whether p or q is the case. But either way, we know that one or the other must be true, and if it's p, then r is true, and it's q, then s is true. So altogether, these three statements imply $r \lor s$.

Thinking metaphorically again, you can imagine that you're either going to board the train at station p or station q. If you board at p, then you'll get off at r. If you board at q, then you'll get off at s. You don't know which one you will board at, but you know you will end up at either station r or station s.

Example Argument

(1)	$(p \,\&\, r) \lor q$
(2)	$(q \to t) \,\&\, r$
(3)	$(p \,\&\, r) \to s$
(4)	$(s \lor t) \lor u$

Proof

(1)	$(p \,\&\, r) \lor q$	Premise
(2)	$(q \to t) \,\&\, r$	Premise
(3)	$(p \,\&\, r) \to s$	Premise
(4)	$q \to t$	Simplification (2)
(5)	$s \lor t$	Constructive Dilemma (1, 3, 4)
(6)	$(s \lor t) \lor u$	Addition (5)

2.10 Disjunctive Syllogism

Simplification

Left rule:			Right rule:		
	(1)	$P \vee Q$		(1)	P
	(2)	$\sim P$		(2)	$\sim Q$
	(3)	Q		(3)	P

Disjunctive Syllogism is simply eliminating a false alternative, or option, from a disjunction. If I am told that either p is the case or q is the case, and then I'm told that p is *not* the case, then there's only one option remaining – q. So, with disjunctive syllogism, we have a disjunction, and we also have the negation of one of the disjuncts, so we can eliminate it and infer the other disjunct. Think of it as a choice between p and q – once one of the choices is eliminated, you can only choose the remaining option.

Example Argument

(1)	$p \,\&\, \sim q$
(2)	$q \vee r$
(3)	$t \rightarrow \sim r$
(4)	$\sim t$

Proof

(1)	$p \,\&\, \sim q$	Premise
(2)	$q \vee r$	Premise
(3)	$t \rightarrow \sim r$	Premise
(4)	$\sim q$	Simplification (1)
(5)	r	Disjunctive Syllogism (2, 4)
(6)	$\sim t$	Modus Tollens (3, 5)

Chapter 3

Rules for Natural Deduction: Rules of Replacement

In this chapter, we will introduce some of the basic rules for natural deduction proofs.

3.1 Double Negation

Double Negation

Left rule:	(1) $\sim\sim P$	Right rule:	(3) P
	(2) P		(4) $\sim\sim P$

Double negation is one of the simplest inference rules in our toolkit. It simply relies on the fact that two negatives "cancel each other out." Think about it this way: Suppose p is true. Negation reverses the value of its argument. So, therefore, if p is true, $\sim p$ is false. Then, if we apply negation again, we simply reverse the value of $\sim p$, so the value of $\sim\sim p$ is true, just like that. Since negation reverses the truth value from true to false, or from false to true, if we apply it twice in a row, then we always end up with the same value as we started with. Therefore, $\sim\sim p$ is equivalent to p. (Similarly, p is always going to have the same truth value as $\sim\sim p$.)

(One thing to note: In English, when we use double negation, we usually try to imply something more than the non-negated statement. For instance, if someone asks, "Is Bob a good singer?" and someone else replies, "Well, he's not *not* a good singer," then they are not exactly saying that Bob *is* a good singer, but they're not denying it either – perhaps they're trying to imply that Bob is a borderline case of a good singer. Either way, this is a matter of the pragmatics of natural language. In propositional logic, we simply assume that two negations cancel each other out.)

Example Argument

(1)	$\sim\sim p \,\&\, \sim\sim q$
(2)	$p \,\&\, q$

Proof

(1)	$\sim\sim p \,\&\, \sim\sim q$	Premise
(2)	$\sim\sim p$	Simplification (1)
(3)	$\sim\sim q$	Simplification (1)
(4)	p	Double Negation (2)
(5)	q	Double Negation (3)
(6)	$p \,\&\, q$	Conjunction (4, 5)

In this proof, we first use Simplification to infer each of the conjuncts of the premise independently (lines 2 and 3). Then, since these are both propositions that have two negations in a row, we use Double Negation to reduce them to p and q independently. Finally, we combine p and q with Conjunction to get $p \,\&\, q$. (*Can you think of a shorter proof for this argument?*)

One important thing to note is that Double Negation only works if there are two negations *in a row* and the inner one is not nested within a more complex statement. For instance, consider:

NOT A VALID PROOF!

(1)	$\sim(\sim p \,\&\, q)$	Premise
(2)	~~$p \,\&\, q$~~	~~Double Negation (1)~~

In this example, I try to use Double Negation to cancel out the two negation symbols at the start of the formula. But this is illegal because the inner negation is actually nested inside of a conjunction: $\sim p \,\&\, q$. Usually, we suppress parentheses when using negation, but if we were more explicit, this would be clear:

$$\sim(\sim p \,\&\, q) = \sim((\sim p) \,\&\, q)$$

Here, you can see that the $\sim$ is actually nested inside of $(\sim p) \,\&\, q$, so we can't apply Double Negation in that way.

3.2 Rules of Replacement

One important thing to note about the rules we have previously covered (Rules 1-8) is that they are only application to *entire formulas*. For instance, the following is *not* O.K.:

(1)	$(p \,\&\, q) \rightarrow r$	Premise
(2)	$p \rightarrow r$	Simplification (1)

In this case, I try to directly apply Simplification to $p \,\&\, q$ in Line 1 to get p, and so I try to convert $(p\&q) \rightarrow r$ to just $p \rightarrow r$. This does not work. Line 1 is a conditional, and the main operator is the $\rightarrow$. So, I cannot apply Simplification because the conjunction is only a subformula.

Intuitively, this is the desired result. For consider the following argument statement:

(1) If it's nice out and I don't have too much work, I'll go to the picnic.

This has the form $(p \,\&\, q) \rightarrow r$. If it were legal to apply Simplification to subformulas, then we could derive $p \rightarrow r$, as in the *bad* proof above. So, we would be able to derive:

(2) If it's nice out, I'll go to the picnic.

But you can't infer (2) from (1). Just because it's nice out doesn't mean the speaker will go to the picnic - what if she has too much work to do? So, the fact that you can't infer (2) from (1) is an illustration of the fact that you can't apply Simplification to subformulas.

Another important observation is that they can only be applied one direction, that is, from the top line(s) to the bottom line. I cannot use Simplification like so (reversing the lines from a previous proof):

NOT A VALID PROOF!

(1)	$p \,\&\, q$	Premise
(2)	~~$(p \,\&\, q) \,\&\, r$~~	~~Simplification (1)~~

These rules are known as **Rules of Implication**. Rules 1-8 are all Rules of Implication:

- Can be applied only to whole formulas (not subformulas).
- Can only be applied in the direction indicated in the rule definition.

What about Double Negation? Let's consider the following argument:

(1)	$\sim\sim p \vee q$
(2)	$(p \vee q) \rightarrow r$
(3)	r

In this example, line 1 is very similar to the antecedent of line 2, but they're not quite the same. If they *were* the same, then we could use Modus Ponens to derive r in line 3.

As it turns out, unlike the previous rules of implication, Double Negation is a rule that we *can* apply to subformulas. So, the following is a valid proof:

(1)	$\sim\sim p \vee q$	Premise
(2)	$(p \vee q) \rightarrow r$	Premise
(3)	$p \vee q$	Double Negation (1)
(4)	r	Modus Ponens (2, 3)

Pay attention to line 3. In this line, we apply Double Negation to convert $\sim\sim p$ to p, but we do so within the disjunction $\sim\sim p \vee q$. This is a known as a **Rule of Replacement**.

How come this is a legal move? Well, unlike the inference from $p \,\&\, qtop$, which only goes in "one direction" (i.e., you can't go from p to $p \,\&\, q$), Double Negation works in "both directions," as you can see in the rule definition. I can go from p to $\sim\sim p$, and from $\sim\sim p$ to p. This means that p and $\sim\sim p$ are **equivalent**. If two statements are equivalent, then you can interchange them whenever they occur, even in subformulas.

In the rest of the chapter, we will look at more rules of replacement.

3.3 Commutativity

Commutativity

Left rule: $\dfrac{(1) \quad p \,\&\, q}{(2) \quad q \,\&\, p}$ Right rule: $\dfrac{(3) \quad p \vee q}{(4) \quad q \vee p}$

This rule is reminiscent of mathematics. Recall that with addition, $7 + 5$ is the same as $5 + 7$. That means that addition is *commutative* (you can move, or "commute," the arguments around). The same is true for disjunctions and conjunctions. Order does not matter when it comes to disjunctions and conjunctions. If $p \,\&\, q$ is true, then obviously $q \,\&\, p$ is also true, because in both cases, both conjuncts must be true. The same argument applies to disjunction. If $p \vee q$ is true, then $q \vee p$ must also be true, since in both cases, at least one of p or q is guaranteed to be true.

Notice, also, that this rule can be applied in "either direction" (i.e., it is a rule of replacement). I can infer $p \,\&\, q$ from $q \,\&\, p$ and I can infer $q \,\&\, p$ from $p \,\&\, q$.

Example Argument

$$\begin{array}{ll} (1) & p \rightarrow (p \,\&\, (q \vee r)) \\ (2) & p \\ \hline (3) & (q \vee r) \,\&\, p \end{array}$$

Proof

(1)	$p \rightarrow (p \,\&\, (q \vee r))$	Premise
(2)	p	Premise
(3)	$p \,\&\, (q \vee r)$	Modus Ponens (1, 2)
(4)	$(q \vee r) \,\&\, p$	Commutativity (3)

3.4 Associativity

Associativity

Left rule:		Right rule:	
(1)	$p \,\&\, (q \,\&\, r)$	(3)	$(p \vee q) \vee r$
(2)	$(p \,\&\, q) \,\&\, r$	(4)	$p \vee (q \vee r)$

Associativity is another term you may recall from mathematics. Associativity applies when there are two operations, and it doesn't matter in which order you apply them. For instance, consider multiplication: $2 \times (3 \times 5) = (2 \times 3) \times 5$. In both cases, if we simplify, we get: $2 \times 15 = 6 \times 5 = 30$. So, when you have a multiplication inside a multiplication, you can switch the order and you will get the same result.

The same is true of conjunction and disjunction. They too have the property of associativity. If I have a conjunction inside a conjunction, like $p \,\&\, (q \,\&\, r)$, then I can move the parentheses around, and it's equivalent to $(p \,\&\, q) \,\&\, r$. This is another rule of replacement, so it also would apply to subformulas as in:

(1)	$(p \,\&\, (q \,\&\, r)) \vee s$
(2)	$((p \,\&\, q) \,\&\, r) \vee s$

Here, $(p \,\&\, (q \,\&\, r))$ is actually a disjunct of the main formula, but I can still apply Associativity.

One crucial thing to note is this rule does *not* apply to a *conjunction* inside a *disjunction*, and vice versa. In other words, if I have a conjunction inside a disjunction, I *cannot* move the parentheses around:

Not a valid proof!

(1)	$p \,\&\, (q \vee r)$	Premise
(2)	~~$(p \,\&\, q) \vee r$~~	~~Associativity (1)~~

There is another rule for these types of situations, but remember that Associativity only works for two of the *same operator* in a row.

Example Argument

(1)	$(p \vee r) \vee s$
(2)	$\sim p$
(3)	$r \vee s$

Proof

(1)	$(p \vee r) \vee s$	Premise
(2)	$\sim p$	Premise
(3)	$p \vee (r \vee s)$	Associativity (1)
(4)	$r \vee s$	Disjunctive Syllogism (2, 3)

3.5 DeMorgan's Law

DeMorgan's Law

Left rule:		Right rule:	
	(1) $\sim(p \,\&\, q)$		(3) $\sim(p \vee q)$
	(2) $\sim p \vee \sim q$		(4) $\sim p \,\&\, \sim q$

DeMorgan's Law (or laws) is a very important rule in logic. At first glance, we have a negation surrounding a conjunction (or a disjunction), and then it looks like we "distribute" the negation across the arguments of the conjunction/disjunction. So, p becomes $\sim p$ and q becomes $\sim q$. However, the crucial thing to note is we *also* have to change the operator, so a conjunction (&) becomes a disjunction ($\vee$), and vice versa.

You can kind of think of this rule as analogous to distributing a negation in arithmetic. For instance, if I have a formula like

$$-(3+5),$$

then that's equivalent to

$$(-3+-5)=-8.$$

So, in this case, I distribute the negation across the addition. DeMorgan's Law is kind of like that, but we also have to remember to change the operator.

If you want an even better mathematical analogy, think of how we manipulate inequalities. Suppose I have an inequality like

$$3n > n+1.$$

Now, if I multiply both sides by -1, I also have to "flip," or reverse the inequality sign. So, we would get:

$$-(3n) < -(n+1),$$

or:

$$-3n < -n-1.$$

In this case, we distribute the negation on both sides of the inequality operator, and we reverse the operator itself. This is akin to how with DeMorgan's Law, we distribute the negation to the arguments and change the operator from $\vee$ to &, or vice versa.

Hopefully, you can also grasp this rule intuitively, as well. Suppose we interpret $\sim p \,\&\, q)$ as, "It's not true that I ate an appetizer and that I ate dessert." The person is saying it's not true that *both* of those things happened (notice, however, they're not saying that *neither* of them happened) – so, that means that at least one of them *didn't* happen. Either the person did not eat an appetizer or they did not eat dessert - or, perhaps they ate neither, but all we know for sure is that *at least* one of "I ate an appetizer" (p) and "I ate dessert" (q) is false. In other words, we can infer $\sim p \vee \sim q$ from $\sim(p \,\&\, q)$. (Note, also, that this rule, or move, can work in either direction – we can also infer $\sim(p \vee q)$ from $\sim p \,\&\, \sim q$.)

Similarly, consider a sentence like $\sim(p \vee q)$, such as, "It's false that I went to that building or ever spoke to that person." If a speaker says this, then they are denying *both* that they went to the building, *and* that they spoke to the person. So, they are claiming $\sim p$ **and** $\sim q$. If you deny a disjunction, then you are asserting the denial of both of the conjunctions, which is why $\sim(p \vee q)$ is equivalent to $\sim p \,\&\, \sim q$.

Study the following proof carefully. It demonstrates a nice interplay between Addition and DeMorgan's Law, and illustrates two applications of DeMorgan's Law.

Example Argument

(1)	$\sim(p \vee q) \vee (p \,\&\, q)$
(2)	$\sim p$
(3)	$\sim q$

Proof

(1)	$\sim(p \vee q) \vee (p \,\&\, q)$	Premise
(2)	$\sim p$	Premise
(3)	$\sim p \vee \sim q$	Addition (2)
(4)	$\sim(p \,\&\, q)$	DeMorgan's Law (3)
(5)	$\sim(p \vee q)$	Disjunctive Syllogism (1, 4)
(6)	$\sim p \,\&\, \sim q$	DeMorgan's Law (5)
(7)	$\sim q$	Simplification (6)

3.6 Distribution

Distribution

Left rule:		Right rule:	
(1)	$p \,\&\, (q \vee r)$	(3)	$p \vee (q \,\&\, r)$
(2)	$(p \,\&\, q) \vee (p \,\&\, r)$	(4)	$(p \vee q) \,\&\, (p \vee r)$

Recall that one of our previous rules, Associativity, allowed us to manipulate formulas with two conjunctions (or disjunctions) "in a row." However, the rule does *not* apply to a mixture of a conjunction and a disjunction, as in $p \,\&\, (q \vee r)$. In these cases, we again do something sort of like distribution in mathematics. In the left-hand side case, we have a disjunction inside a conjunction, so we distribute the conjunction across the arguments of the disjunction, and then combine the two results with a disjunction. On the right-hand side, we do the same thing, but with conjunction and disjunction signs switched.

Think of how we distribute multiplication across an addition statement, e.g.:

$$3 \times (4 + 5).$$

This is equivalent to

$$3 \times (9) = 27.$$

But, we could also distribute the multiplication sign first, as in:

$$3 \times (4+5) = (3 \times 4) + (3 \times 5)$$
$$(3 \times 4) + (3 \times 5) = 12 + 15$$
$$12 + 15 = 27.$$

The same pattern applies when we distribute a conjunction across a disjunction, or a disjunction across a conjunction.

This rule can also be grasped intuitively. Consider the left-hand side. If we have $p \,\&\, (q \vee r)$, then we know for sure that p is true, and we know that either q or r is true. So, there's two possibilities: Either p *and* q are true together, or p *and* r are true together. We don't know which one, so we form a disjunction between those two possibilities, i.e., $(p \,\&\, q) \vee (p \,\&\, r)$.

Example Argument

(1)	p
(2)	q
(3)	$(p \,\&\, q) \vee (p \,\&\, r)$

Proof

(1)	p	Premise
(2)	q	Premise
(3)	$q \vee r$	Addition (2)
(4)	$p \,\&\, (q \vee r)$	Conjunction (1, 3)
(5)	$(p \,\&\, q) \vee (p \,\&\, r)$	Distribution (4)

3.7 Transposition

Transposition

(1)	$p \rightarrow q$
(2)	$\sim q \rightarrow \sim p$

Transposition can be a very handy rule, and should be easy to grasp by now. When we apply Transposition to a conditional, we (i) reverse the order of the arguments; and, (ii) negate both sides. Note that $\sim q \rightarrow \sim p$ is also known as the **contrapositive** of $p \rightarrow q$.

Why is this move allowed? Well, think about a rule we've already covered, Modus Tollens:

(1)	$p \rightarrow q$
(2)	$\sim q$
(3)	$\sim p$

Modus Tollens tells us that if we have a conditional, and we also can infer the negation of the consequent of the conditional, then we can infer the negation of the antecedent. Transposition relies on this same fact. We start with a conditional ($p \to q$). And then we reason that if we had the negation of the consequent ($\sim q$), *then* we could guarantee the negation of the antecedent ($\sim q$). So, whereas Modus Tollens takes a conditional *and* the negation of the consequent, with Transposition, we turn one conditional into another by saying if the consequent is false (which we don't know yet), then the antecedent is false as well.

Example Argument

(1)	$(\sim r \to \sim p) \to (r \to s)$
(2)	$p \to q$
(3)	$q \to r$
(4)	$p \to s$

Proof

(1)	$(\sim r \to \sim p) \to (r \to s)$	Premise
(2)	$p \to q$	Premise
(3)	$q \to r$	Premise
(4)	$p \to r$	Hypothetical Syllogism (2, 3)
(5)	$\sim r \to \sim p$	Transposition (4)
(6)	$r \to s$	Modus Ponens (1, 5)
(7)	$p \to s$	Hypothetical Syllogism (4, 6)

3.8 Exportation

Exportation	
(1)	$(p \,\&\, q) \to r$
(2)	$p \to (q \to r)$

This rule can be a bit tricky to remember, and it might look a little strange at first. Actually, however, we can try to make it very intuitive. Consider the top line - this asserts that *if* p and q are true, then r will be true. So, if p and q are both true, then r is true. Now, let's look at the second line. What if we start with just p? If just p is true, then we can't infer r, because we are told that p and q *together* imply r in line 1. However, we can reason as follows: *if* p is true, *then if* q is true as well, then we know r is true. So, in a way, we take the conjunction $p \,\&\, q$ and break it down into pieces - first, *if* we get p, then we are halfway there towards $p \,\&\, q$, so *if* we get p, then *if* we also get q, we will get r. Symbolically: $p \to (q \to r)$.

Example Argument

(1)	$\sim p$
(2)	q
(3)	$\sim p \to (q \to (r \vee p))$
(4)	r

Proof

(1)	$\sim p$	Premise
(2)	q	Premise
(3)	$\sim p \to (q \to (r \vee p))$	Premise
(4)	$\sim p \,\&\, q$	Conjunction (1, 2)
(5)	$(\sim p \,\&\, q) - > (r \vee p)$	Exportation (3)
(6)	$r \vee p$	Modus Ponens (4, 5)
(7)	r	Disjunctive Syllogism (1, 6)

3.9 Tautology

Tautology

(1)	$p \vee p$
(2)	p

Tautology is a rule that perhaps does not come up as frequently as some other rules, since it's unusual to find yourself with a line of the proof with the form $p \vee p$. However, if it does arise, the intuition behind this inference rule should be clear. Normally, if we have a disjunction, like $p \vee q$, we can't simply eliminate one of the disjuncts and infer p or infer q - that's because we don't know which of the two is true! But what if the disjunction is of the form $p \vee p$? Well, let's consider both "possibilities." If the left hand side happens to be true, then p is true. If the right-hand side is true, then p is also true. We know that at least one of the sides has to be true, and in either case we know p is true, so therefore we can directly infer p from $p \vee p$.

Example Argument

(1)	$(\sim\sim p \to q) \vee (\sim q \to \sim p)$
(2)	$p \to q$

Proof

(1)	$(\sim\sim p \to q) \vee (\sim q \to \sim p)$	Premise
(2)	$(p \to q) \vee (\sim q \to \sim p)$	Double Negation (1)
(3)	$(p \to q) \vee (p \to q)$	Transposition (2)
(4)	$p \to q$	Tautology (3)

3.10 Material Implication

Material Implication	
(1)	$p \to q$
(2)	$\sim p \vee q$

Material Implication, or Material Conditional, is another essential rule in logic. It shows us that there is a fundamental equivalence between a conditional and a disjunction. But how are these two related? Why are they equivalent?

One way to understand the equivalence is to construct a proof using rules we've learned already. Think about what $p \to q$ means. Recall that a conditional is only false when the antecedent is true and the conditional is false. So to assert $p \to q$ is to assert that it's *not* the case that $p \,\&\, \sim q$ (since that would make the conditional false). So, intuitively, we can think of $p \to q$ as equivalent to $\sim(p \,\&\, \sim q)$. Then, we can show how to reach $\sim p \vee q$:

(1)	$\sim(p \,\&\, \sim q)$	Premise
(2)	$\sim p \vee \sim\sim q$	DeMorgan's Law (1)
(3)	$\sim p \vee q$	Double Negation (2)

So, if we think of a conditional in this way, we can see why $p \to q$ is equivalent to $\sim p \vee q$.

Here's another way to think about it. Recall that a conditional is true whenever its antecedent is false. So, let's take $p \to q$. Well, p could either be true or false. If p is false, then the conditional is true, so we're all good. But if p is *true*, then for the conditional to be true, q has to be true. So, *either* p is false, *or* p is true and q is also true. That is, either $\sim p$ or $p \,\&\, q$, which reduces to $\sim p \vee q$, as shown here:

(1)	$\sim p \vee (p \,\&\, q)$	Premise
(2)	$(\sim p \vee p) \,\&\, (\sim p \vee q)$	Distribution (1)
(3)	$\sim p \vee q$	Simplification (2)

The fact that a conditional can be converted to a disjunction (and vice versa) can be a very useful tactic in many proofs.

Example Argument

(1)	$p \vee q$
(2)	$(\sim p \to q) \to (r \,\&\, s)$
(3)	$s \,\&\, r$

Proof

(1)	$p \vee q$	Premise
(2)	$(\sim p \to q) \to (r \,\&\, s)$	Premise
(3)	$\sim p \to q$	Material Implication (1)
(4)	$r \,\&\, s$	Modus Ponens (2, 3)
(5)	$s \,\&\, r$	Commutativity (4)

Note that we can't move *directly* from line 3 to line 5. That is, when we apply Modus Ponens using lines 2 and 3, we can derive the consequent of line 2 – $r \,\&\, s$ – but, we have to keep the order of the consequent the same. We can't, in the same move, switch $r \,\&\, s$ to $s \,\&\, r$. So, to be precise, we *first* derive $r \,\&\, s$, and then we can independently apply Commutativity to derive $s \,\&\, r$ (line 5). Make sure not to skip steps in your proof, or combine multiple moves into one, even if it seems really obvious.

3.11 Material Equivalence

Material Equivalence

Left rule:		Right rule:	
(1)	$p \leftrightarrow q$	(3)	$p \leftrightarrow q$
(2)	$(p \to q) \,\&\, (q \to p)$	(4)	$(p \,\&\, q) \vee (\sim p \,\&\, \sim q)$

Looking at the left-hand side, this rule is about taking a biconditional and breaking it up into a conjunction of two conditionals. It essentially formalizes the meaning of the biconditional itself. A biconditional like

$$p \leftrightarrow q$$

is equivalent to

$$p \to q \text{ and } q \to p.$$

(In fact, using the "arrow" notation for conditionals and biconditionals makes this obvious, since we see the arrow pointing from p to q, and also from q back to p.) So, a biconditional is equivalent to a conjunction of conditionals from each side to the other.

Now, let's look at the right-hand side. This looks very different, but again, it closely mirrors the basic definition of the biconditional. Thinking about the truth table for a biconditional, recall that the key requirement for a biconditional to be true is that both sides have the *same value* (i.e., both true, or both false). So, a biconditional like

$$p \leftrightarrow q$$

is equivalent to saying *either* p and q are both true ($p \,\&\, q$), or p and q are both false ($\sim p \,\&\, \sim q$). Putting those together with a disjunction, we get

$$(p \,\&\, q) \vee (\sim p \,\&\, \sim q).$$

Example Argument

(1)	$(p \to q) \leftrightarrow (r \to s)$
(2)	$\sim p$
(3)	$\sim s$
(4)	$\sim r$

Proof

(1)	$(p \to q) \leftrightarrow (r \to s)$	Premise
(2)	$\sim p$	Premise
(3)	$\sim s$	Premise
(4)	$\sim p \vee q$	Addition (2)
(5)	$p \to q$	Material Implication (4)
(6)	$((p \to q) \to (r \to s))\&$ $((r \to s) \to (p \to q))$	Material Equivalence (1)
(7)	$(p \to q) \to (r \to s)$	Simplification (6)
(8)	$r \to s$	Modus Ponens (5, 7)
(9)	$\sim r$	Modus Tollens (3, 8)

Chapter 4

Advanced Proof Techniques

4.1 Indirect Proof

Now that we've mastered all the rules for natural deduction that we're going to cover, let's see how we can use them to prove whether a statement is a tautology, also known as a logical truth. Recall that a tautology is a statement that is true in every possible set of circumstances. Tautologies are important because they are special statements that are guaranteed to be true no matter what.

For instance, consider a proposition like $p \vee \sim p$, "Either it's raining or it's not raining." This statement is a tautology - it's always true, true no matter what. At any given time, at any place, it either *is* raining or it's *not* raining.

(In fact, you might think that there are some "in-between" states where it's not clear whether it's raining or not. As mentioned previously, we ignore these vagaries in our discussion of classical propositional logic, and we assume that every proposition is either true or false.)

We've seen already how to use natural deduction inference rules to show whether an argument is valid or not - you have to provide a proof from the premises to the conclusion. But how can we show whether a statement is a tautology? A tautology is not an argument with premises and conclusion - in a way, we can think of it as just a conclusion, with no premises. If you can prove that statement/conclusion without using any premises, then clearly that statement must be true no matter what.

Since a tautology is a statement that is true in every possible circumstance, then the negation of a tautology would be a statement that is true in *no* possible circumstances (all rows in the truth table would flip from "`TRUE`" to "`FALSE`"). In other words, negating a logical truth yields a situation that is *impossible*. But what is an "impossible" situation? Well, any possible combination of truth values for the atomic variables of a proposition represents a possible situation. Every variable, or state of affairs, is assigned either True or False. What would be *impossible* is if a certain proposition, say p, were assigned *both* `TRUE` and `FALSE`! Consider the sentence "It's raining and it's not raining." That does not describe a possible state of affairs - it has to be one or the other, and not both. Whenever we have a situation where we have asserted a proposition and also its negation, then we have derived a **contradiction**.

If we are able to derive a contradiction in our proof, that is, if we are able to infer on two separate lines both p (or any proposition) and its negation, then we know that we have started with a proposition that entails a contradiction.

So, putting this all together: we know how to identify a contradiction - we apply our inference rules and try to infer two lines that are the negation of one another. We also know that the negation of a tautology is a contradiction (or a statement that is always false). So, if we want to know whether a statement is a tautology, (1) we start by asserting its *negation* as a premise (which, by hypothesis, should be a contradiction), and then (2) we apply our inference rules to try to derive a direct contradiction (i.e., two lines that are the direct negation of each other).

Let's see how this works in practice. Suppose we want to prove that the following is a tautology:

(Z) $p \rightarrow ((p \rightarrow q) \rightarrow q)$

Proof

Begin by asserting the negation of Z:

(1)	$(\sim(p \rightarrow ((p \rightarrow q) \rightarrow q))$	Assumption
(2)	$\sim(\sim p \vee ((p \rightarrow q) \rightarrow q))$	Material Implication (1)
(3)	$\sim\sim p \,\&\, \sim((p \rightarrow q) \rightarrow q)$	DeMorgan's Law (2)
(4)	$p \,\&\, \sim((p \rightarrow q) \rightarrow q)$	Double Negation (3)
(5)	p	Simplification (4)
(6)	$\sim((p \rightarrow q) \rightarrow q)$	Simplification (4)
(7)	$\sim(\sim(p \rightarrow q) \vee q)$	Material Implication (6)
(8)	$\sim\sim(p \rightarrow q) \,\&\, \sim q$	DeMorgan's Law (7)
(9)	$\sim\sim(p \rightarrow q)$	Simplification (8)
(10)	$p \rightarrow q$	Double Negation (9)
(11)	$\sim q$	Simplification (8)
(12)	$\sim p$	Modus Tollens (10, 11)
(13)	$p \,\&\, \sim p$	Conjunction (5, 12)
(14)	$\sim\sim(p \rightarrow ((p \rightarrow q) \rightarrow q))$	Indirect Proof (1, 13)
(15)	$p \rightarrow ((p \rightarrow q) \rightarrow q)$	Double Negation (14)

Study this proof carefully to follow the reasoning. Note that we begin by assuming the negation of the statement we are trying to prove (Z). Then we apply a series of rules to derive p on line 5, and then apply more inference rules to derive $\sim p$ on line 12. Now, we have found a contradiction, so we combine them on line 13 using Conjunction. This is a directly contradictory statement. Now that we have proven a contradiction, we can infer the *negation of line 1* via Indirect Proof (citing also the line containing the contradiction). To be clear: when you prove a contradiction, you have proven *the negation of the premises or assumptions you started with.* So, line 14 is the negation of line 1. Then, we simply eliminate the leading double negation, and we have proven (Z) by showing that the negation of (Z) is inconsistent.

This method is known as **indirect proof** because we are proving a statement *indirectly*, by showing that its negation leads to contradiction.

4.2 Arguments

We have seen how to use indirect proof to judge whether a statement is a tautology. This method can also be used to judge whether an argument is valid. How would we do this? Let's recall what a valid argument is. If an argument is valid, then we are guaranteed that if the premises are true, then the conclusion is definitely true. That is, if an argument is valid, it's *impossible* for the premises to be true and the conclusion false. So, we can once again use indirect proof to test whether such a situation is impossible. That is, to apply indirect proof to judge whether an argument is valid, we (1) assert all the premises and assume the negation of the conclusion, and then (2) we use the inference rules to try to derive a contradiction. If we find a contradiction, then we know the combination of the premises and the *negation* of the conclusion is impossible, and therefore that the premises guarantee that the conclusion is true; in other words, we know the argument is valid.

Let's see this in action.

Example Argument

(1)	p
(2)	$(p \vee \sim q) \rightarrow \sim(p \vee r)$
(3)	$\sim r$

Proof

(1)	p	Premise
(2)	$(p \vee \sim q) \rightarrow \sim(p \vee r)$	Premise
(3)	$\sim\sim r$	Assumption
(4)	$p \vee \sim q$	Addition (1)
(5)	$\sim(p \vee r)$	Modus Ponens (2, 4)
(6)	$\sim p \,\&\, \sim r$	DeMorgan's Law (5)
(7)	$\sim r$	Simplification (6)
(8)	$\sim r \,\&\, \sim\sim r$	Conjunction (3, 7)
(9)	$\sim r$	Indirect Proof (3, 8)

Here, we start with the premises of our proof; then, we make an additional assumption ($\sim\sim r$), which is the negation of the conclusion that we want to reach ($\sim r$). Then we apply some more rules until we have derived a contradiction:

$$r \,\&\, \sim r.$$

Since we were able to derive a contradiction, we know that the combination of the premises and the negation of the conclusion is contradictory, and hence the argument is valid.

4.3 Within Arguments

The method of indirect proof can also be used *within* the course of a proof. If you're in the middle of a proof, you can start a "branch" of that proof by making a new assumption, say p. Unlike other lines in the proof, which have to be justified based on one of the inference rules, making a new assumption does not have to be justified. Like a premise, it's just asserted. However, since it's not one of the original premises of the argument, it can't directly be used to derive the conclusion. However, if you're able to derive a *contradiction* after making that new assumption, that means that p is not possible relative to the premises of the argument, so we can add $\sim p$ to the proof. Let's look at an example. In this case, we will introduce a more explicit notation for indirect proof. Whenever you make an additional assumption, we will indent the propositions, to indicate that we are on a separate "branch" of the argument.

Proof

(1)	$r \to \sim p$
(2)	r
(3)	$p \vee q$
(4)	$q \vee s$

(1)	$r \to \sim p$	Premise
(2)	r	Premise
(3)	$p \vee q$	Premise
(4)	$\sim p$	Modus Ponens (1, 2)
(5)	$\sim q$	Assumption
(6)	p	Disjunctive Syllogism (3, 5)
(7)	$p \,\&\, \sim p$	Conjunction (4, 6)
(8)	q	Indirect Proof (5, 7)
(9)	$q \vee s$	Addition (9)

This proof is a little silly, because a clever proof-solver would directly infer q from lines (3 and 4) using Disjunctive Syllogism. But here we illustrate a more roundabout technique. On line 5, we simply *assume* $\sim q$ to be true (notice that no lines are cited, as there is no justification for an assumption). Now, until we have applied Indirect Proof, everything that we infer can only be assumed to be true given $\sim q$. Hence, we indent these lines. On line 7, we have derived a direct contradiction. Then, on line 8, we "break out" of the branch that we started by assuming $\sim q$, and since we found a contradiction after making that assumption, we know that that assumption must be false, hence we infer the negation of line 5. Eventually, we reach our goal of $q \vee s$.

4.4 Conditional Proof

Consider the following proof:

(1)	p
(2)	$(p \,\&\, q) \rightarrow r$
(3)	$q \rightarrow r$

It's possible to solve this proof using the rules that we have covered already. For instance, here is an example:

(1)	p	Premise
(2)	$(p \,\&\, q) \rightarrow r$	Premise
(3)	$p \rightarrow (q \rightarrow r)$	Exportation (2)
(4)	$q \rightarrow r$	Modus Ponens (1, 3)

That's a relatively simple proof. But let's think through the argument intuitively. In line 1, we're told that p is the case. Then, in line 2, we're told that *if* p and q are *both* true, then r is true. Now, we only know p for sure, so we can't infer r. But we know if that if q *were* true, then r would be true as well (since $p \,\&\, q$ would be true). So, in that sense, q implies r.

There is another technique of natural deduction proof that we can use to model this kind of thinking. It is known as **conditional proof**. As in indirect proof, in conditional proof, we add an additional assumption to the proof. But rather than deriving a contradiction in order to refute that assumption, we use it to derive a conditional: if the assumption (whatever it is) is true, *then* some other statement is true.

Let's look at another solution to the above proof, this time using conditional proof:

(1)	p	Premise
(2)	$(p \,\&\, q) \rightarrow r$	Premise
(3)	q	Assumption
(4)	$p \,\&\, q$	Conjunction (1, 3)
(5)	r	Modus Ponens (2, 4)
(6)	$q \rightarrow r$	Conditional Proof (3, 5)

Observe that on line 3, we make a new assumption, and again visually indicate this by starting a new "branch." Now, we are allowed to use q in the proof, but everything we derive will be dependent on q. In two steps, we reach r on line 5. So, what does this mean? It means *if* we assume q to be true, *then* we can derive r. In other words, to put it in symbols, $p \rightarrow r$.

So, conditional proof is very powerful. If you are trying to derive a conditional statement, you can make an assumption declaring the antecedent to be true, and then see if you can derive the consequent with that new information. If you can, then you know that the antecedent implies the consequent. Moving off of the branch as we go from line 5 to line 6 is also known as "discharging" the assumption. You must discharge any new assumptions (i.e., not part of the stated premises) that you add to your proof before reaching the final conclusion.

That's because with conditional proof, we're not necessarily proving something to be positively true - we're showing that something is true *conditionally.* Then, we have to discharge that assumption by deriving a conditional statement.

Let's look at one more example:

Example Argument

(1)	$(p \to q) \to r$
(2)	$\sim p \lor s$
(3)	$s \to q$
(4)	r

Proof

(1)	$(p \to q) \to r$	Premise
(2)	$\sim p \lor s$	Premise
(3)	$s \to q$	Premise
(4)	p	Assumption
(5)	s	Disjunctive Syllogism (2, 4)
(6)	q	Modus Ponens (3, 5)
(7)	$p \to q$	Conditional Proof (4, 6)
(8)	r	Modus Ponens (1, 7)

In this proof, we assume p in line 4, then use that information to derive q, and then discharge our assumption p in line 7, which allows us to reach the conclusion.